Power Transmission and Timing

A frequent task which arises in design is to transfer power from a driving source (usually a rotational source) and a driven output. This task is called power transmission.

A task related to power transmission is timing. In the internal combustion engine, fuel and air delivery to the piston cylinder and the spark had to be timed mechanically. A timing belt or chain was used for this purpose.

In modern systems, sensors usually determine the timing information. However, sometimes these sensors must be mechanically coupled to the driving source, usually through the same mechanical mechanisms of days gone by.

The difference between power transmission and timing are slight.

In power transmission, the forces are larger than in timing, and the methods for shaft coupling require more robust design techniques.

Timing requires greater accuracy. This usually translates into less back-lash and more expensive components.

Methods for achieving power transmission include:

- **Gears**
- **Belts and Pulleys**
- **Sprocket and Chain**
- **Timing Belt and Gears**

Belts and Pulleys (Machinery Handbook pg. 2372-2409)

The basic concepts of power transmission are best illustrated through the belt and pulley.

Examples of the belt and pulley system:

- **Power distribution belts in an automobile (e.g. the belt that drives the A/C, water pump, etc.)**
- **Coupling between motor and spindle in a milling machine or band-saw**

Belt slip determines the maximum power transmission. V-belts alleviate this problem by increasing the effective surface area over which contact is maintained.

Belt and pulley must be pretensioned. This adds to bearing loads in a system.

Belts wear over time and must be replaced. Improvements in belt technology and automatic tensioners have reduced this problem.

Although machinery handbook gives design tables and equations for belt calculations, the designer should always refer to manufacturer design information where it exists.

Different Pulley Diameters

One use for belts is to apply an increase in torque between the driving element and the driven element.

Consider two pulleys driven by a belt. The first pulley has radius r_1 and the second pulley has a different radius, r_2. Let us perform some rudimentary analysis on this system.

Assume the first pulley turns through an angle, θ_1. The belt moves through an arc length, $s_1 = r_1\theta_1$. The second pulley must move through this same length, assuming the belt does not bunch or slip. Hence, $s_1 = r_2\theta_2$ and $r_1\theta_1 = r_2\theta_2$.

Differentiating this equation yields $r_1\dot{\theta}_1 = r_2\dot{\theta}_2$ which may be rearranged to determine the gear ratio, $\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{r_2}{r_1}$.

In words, the ratio of output speed to input speed is the inverse ratio of the pulley radii.

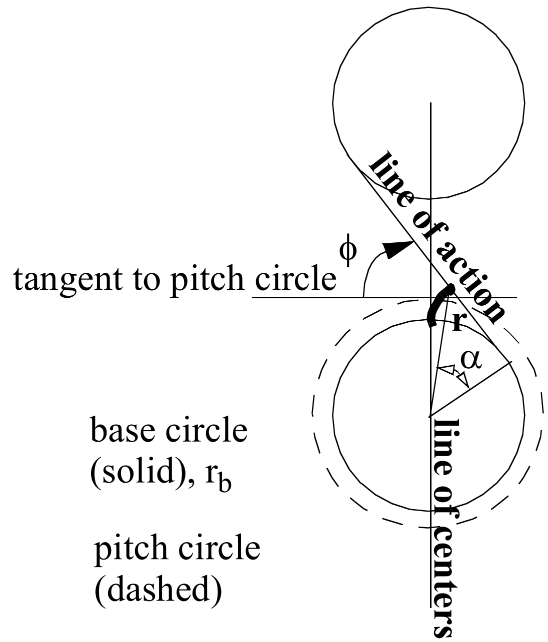
This provides a powerful tool for matching conditions. If the driving torque is fixed, any set of pulleys can be devised which give an output speed in any range desired.

NOTE: the direction of rotation is the same for driving pulley and driven pulley. The separation between pulley centers is mostly unrestricted.

Gears

The problems with pulleys include slip (which limits maximum power transmission), wear (which limits maximum life), load on bearings (which increases losses), and set up time. These problems are solved by gears.

The gear tooth form is created by an involute curve. Pretend that a string is rolled up on a cylinder whose diameter is called the <u>base diameter</u>. As the string is unrolled (while remaining taught) it traces out a curve in space, called the involute curve. This curve forms the tooth form.



If two cylinders of different diameters were to maintain string contact while the string were unwinding on one circle and winding onto the other, the rate of winding/unwinding would be uniform and not depend on center distance.

The involute function is $\mathrm{inv}(\alpha) = \tan(\alpha) - \alpha$**, where** α **is the angle through which the string is unwrapped. The radius of the tooth form is given by** $r = \dfrac{r_b}{\cos(\alpha)}$**. As** α **is varied, the line traces out the tooth form. When** α **reaches the pitch circle, it takes on the value of the pressure angle,** ϕ**. The pressure angle is defined as the angle between the line-of-action and a perpendicular to the line of centers.**

The equation relating pressure angle and base circle diameter is $D_b = D\cos\phi$ **where D is the pitch diameter.**

The pitch diameter relates the number of teeth on a gear and the diametral pitch.

Gear Terminology

The diametral pitch of a gear determines how many teeth per inch a gear contains.

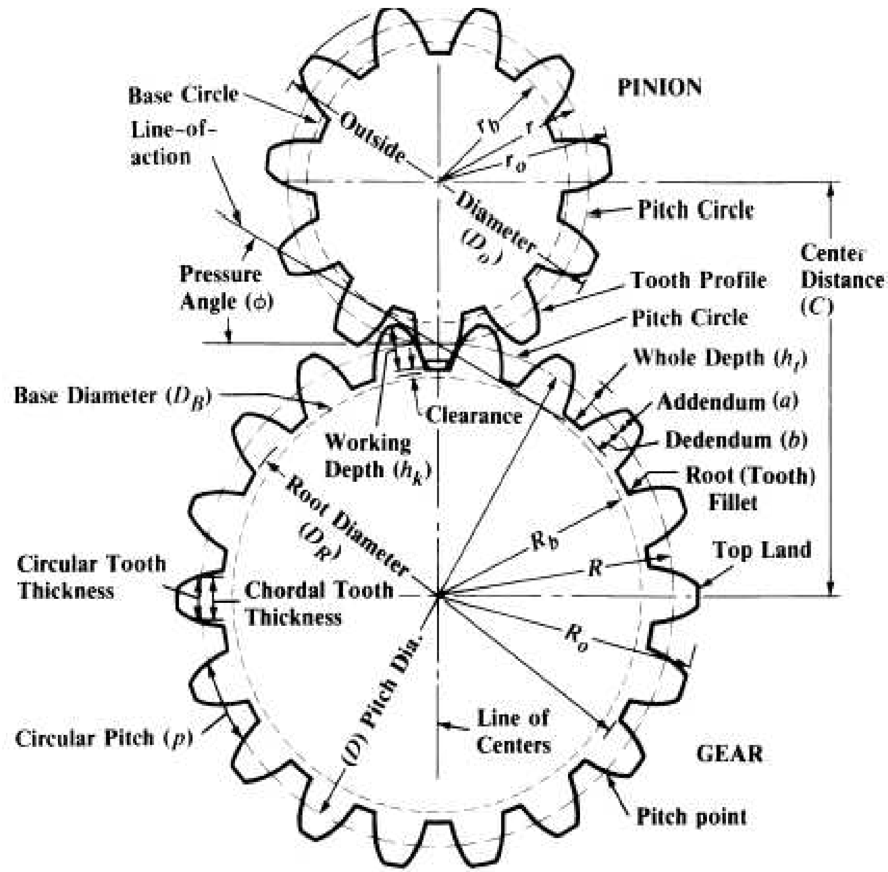
For instance, 32 pitch means that a gear has 32 teeth per inch. Note that the linear distance in this measurement is one of arc length for spur gears.

For metric gears, the same concept is called module. However, module is the inverse concept. It is the number of millimeters per tooth.

For instance, a 0.7 module means that one tooth occupies 0.7 mm of arc length.

Since module and pitch are linked to a specific unit, the conversion is:

$$P_d = (25.4\text{mm/inch})(m^{-1}). \quad \textbf{(1)}$$



Reprinted from Machinery's Handbook, 26th ed.

The pitch diameter is the circle along which gear meshing occurs. As two gears whose centers are correctly placed mesh, it is as if two cylinders, each with the pitch diameter of its gear, were rolling on each other *without slip*. Pitch diameter can be calculated from pitch and number of teeth (Z) by $D = Z/P_d = mZ$.

Geometry of Gears

In order for gears to roll properly along each other, their centers must be spaced at the distance:

$$L = \frac{(D_1 + D_2)}{2} = \frac{(Z_1 + Z_2)}{2P_d} = \frac{m(Z_1 + Z_2)}{2}. \qquad \textbf{(1)}$$

Forces between gear teeth will be directed along the <u>pressure angle</u>. In a well designed gear, the force will be mostly perpendicular to the gear tooth face. This means that most of the force will be directed towards turning the gear and little will be pushing against the bearing support.

Pressure angles for readily available gears (English or metric) are usually quantized into the following bins, $14\frac{1}{2}°$**,** $20°$**,** $25°$**. Custom pressure angles can be made. However, this is done at the expense of much design time and design of tooling. Although it is still readily available,** $14\frac{1}{2}°$ **is being phased out and should not be used unless absolutely necessary.**

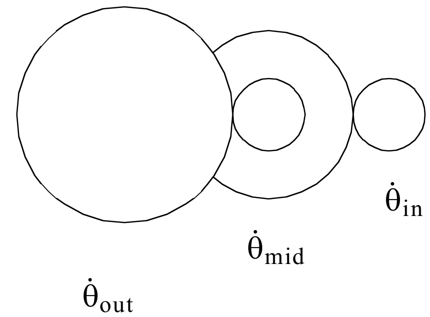
The larger the pressure angle, the thicker the tooth and the more normal the force. This results in a lessening of the unused force and strenthens the tooth at the same time.

For larger pressure angle gears, the smallest number of teeth on a gear is more than for smaller pressure angle gears. E.g., the smallest gear for a $25°$ **PA might be 10, whereas for** $14\frac{1}{2}°$ **PA, it might be eight.**

Larger pressure angle gears have a larger back-lash than smaller pressure angle gears.

Gear Train Calculations

The classic spur gear train involves two geometric rules.



1. Gears which share a common shaft have the same rotational speed.

2. Meshing gears have the speed ratio $\frac{\dot{\theta}_{out}}{\dot{\theta}_{in}} = -\left(\frac{D_{in}}{D_{out}}\right) = -\left(\frac{Z_{in}}{Z_{out}}\right)$**, where D is the pitch diameter and Z is the number of teeth on a gear. This formula comes from the same geometric consideration as two pulleys.**

Note the negative sign in this formulas. It indicates that direction is reversed for each gear stage. This is a difference from pulleys, where the direction remains the same. The normal use is to have smaller gears mesh with larger gears.

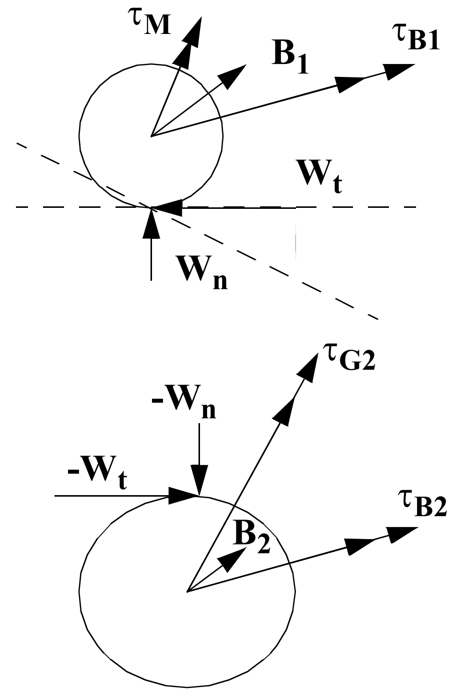
A large gear and a small gear at a middle stage will share a shaft. These two gears have the same angular speed.

Hence, for a two stage gear train $\frac{\dot{\theta}_{mid}}{\dot{\theta}_{in}} = -\left(\frac{Z_{in}}{Z_1}\right)$**,** $\frac{\dot{\theta}_{out}}{\dot{\theta}_{mid}} = -\left(\frac{Z_2}{Z_{out}}\right)$**, where** Z_1 **and** Z_2 **share a common shaft. This formula can be replicated *ad infinitum*.**

Thus, $\frac{\dot{\theta}_{out}}{\dot{\theta}_{in}} = \left(\frac{Z_2}{Z_{out}}\right)\left(\frac{Z_{in}}{Z_1}\right)$**.**

Forces in Spur Gears

Assume that gears are moving at constant velocity. During starting and stopping, there will be inertial "forces" due to the gear's moment of inertia. This can be accommodated by applying a safety factor. Except for dramatic stoppages, such as when a gear train is locked up due to an external failure, angular accelerations will be moderate. Small gears have small moments of inertia. The product of these two factors, except under extreme circumstances, will be negligible when compared with the actual moments. In other words, except when locking up the gear system (an unusual circumstance), the dynamic aspect of gears can be neglected for purposes of design.



Even though the gear system may be moving in space, the resulting linear accelerations and Coriolis terms will be very small compared to both the forces and moments encountered in a gear system.

Thus, a static analysis will be fairly accurate under all circumstances. In fact, the tolerances in materials will affect the calculations more than accelerations.

Two components of forces are considered:

- forces tangent to the pitch circle, which transmit motion from one gear to the next
- normal forces, which serve no useful purpose, but occur because the pressure angle is not "perfect"

Consider the free body diagram of the gears and note the implied direction of W_t.

Static Analysis of Spur Gears

This is a planar problem. So, we can sum forces in the x and y directions, and sum moments in the z direction. Note: the y direction is always aligned with the line of action and the x direction is aligned perpendicular to this.

This yields six equations for the two gears.

$$B_{1x} = W_t\ ,\ B_{1y} = -W_n,\ B_{2x} = -W_t,\ B_{2y} = W_n \tag{1}$$

$$\tau_M + \tau_{B1} - r_1 W_t = 0,\ \tau_{G2} + \tau_{B2} + r_2 W_t = 0. \tag{2}$$

Unless you prepare a model for bearing torques (ie as a function of angular speed or bearing forces) it is best to neglect them in the analysis and develop an experiential model for the friction. Hence,

$$\frac{\tau_{G2}}{\tau_M} = -\frac{r_2}{r_1} = -\frac{Z_2}{Z_1}. \tag{3}$$

For spur gears, I normally use a fudge factor of 0.9. For planetary gears, 0.8. And for worm gears, 0.7 to account for friction at the bearings. This might also be a function of the actual bearings. I.e., better bearings will have lower frictional losses.

$$\frac{\tau_M}{\tau_{G2}} = (-f)\frac{Z_2}{Z_1} \tag{4}$$

Calculating the Forces

From equation 2 on the previous slide (neglecting friction), the tangent force is:

$$W_t = \left(\frac{\tau_M}{r_1}\right) \quad \textbf{(1)}$$

Since this force will be used in failure calculations, by neglecting the friction torque, this force will be larger than the actual force. However, this is erring in the direction of conservatism for failure criteria and is a good thing.

Because the force acts along the pressure line, the horizontal component of the force is:

$$W_n = W_t \tan\phi \quad \textbf{(2)}$$

It is often useful to treat gears through the power that they transmit. This occurs because power encapsulates both the torque and the velocity.

Consider the work rate done at the point of contact. This is the force acting at that point multiplied by the velocity of the point. This is actually a dot product and, since the entire velocity is tangentially directed at the circumference of a circle, only the tangential force enters into the equation:

$$P = W_t v_t = W_t r_1 \omega_1 = W_t r_2 \omega_2 = \tau_M \omega_1 . \quad \textbf{(3)}$$

Noting that, for DC motors, $\tau_M = T_{stall}\left(1 - \frac{\omega}{\omega_{no\ load}}\right)$**, the power at the first stage is** $P = T_{stall}\omega_1\left(1 - \frac{\omega_1}{\omega_{no\ load}}\right)$**.**

Internal Gearing

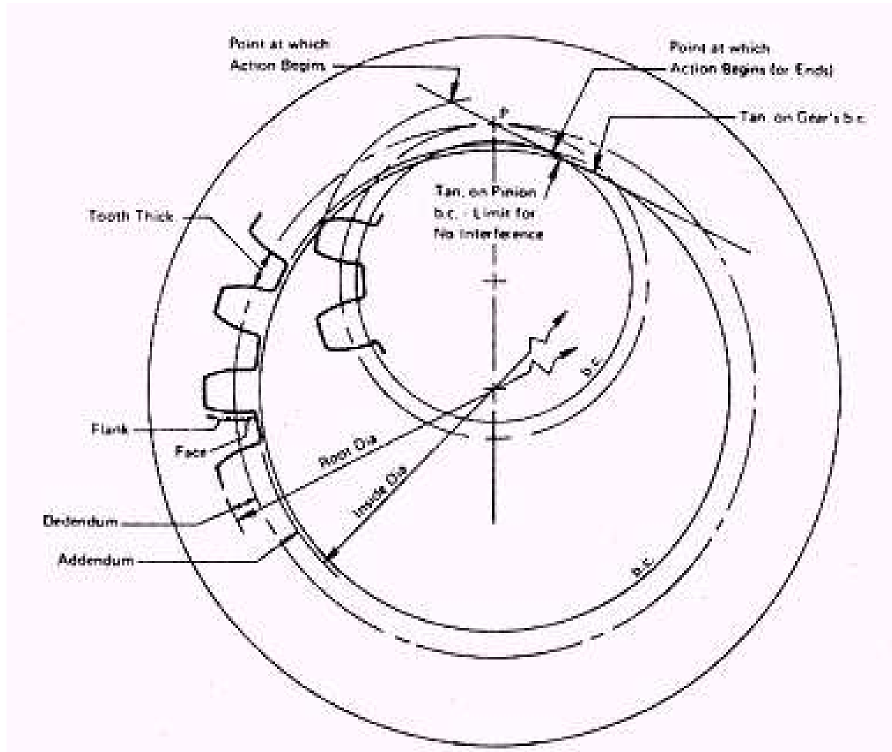
An internal gear is a spur gear (or helical gear if you have some extra cash) turned inside out.

The addendum and dedendum are reversed, and usually the internal gear is cut deeper and the pinion tooth is lengthened.

Standard spur gears will mate with internal gears. However, the margin for error (ie with number of teeth) must be considered more carefully than with conventional spur gear meshes.

Do not make the pinion too large or interference will result.

In internal gear meshes, the pinion rolls around the inside of the internal gear (relatively speaking).



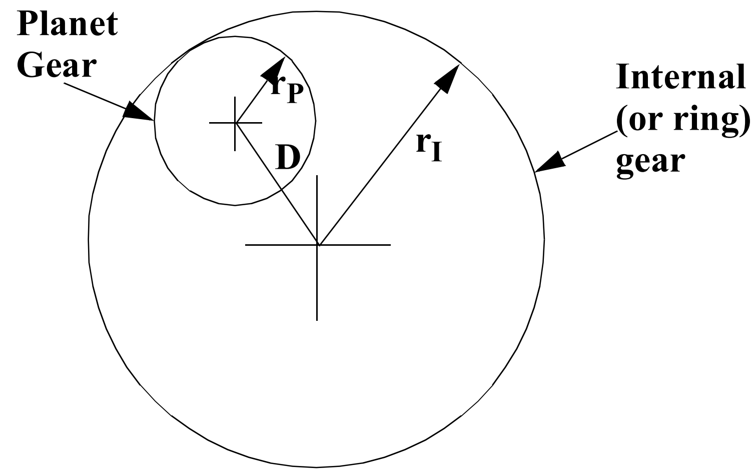
Internal Gear. From Stock Drive Products Design Manual

In most cases, either the internal gear will be pinned to avoid rotation or the center of the pinion will be pinned to avoid rotation.

In the first case, the pinion's axis will rotate. In the second case, the internal gear will rotate about its axis.

Center Distance for Internal Gears

The geometry of internal gears involves a small circle inscribed in a larger circle.



In order for the two circles to be tangent, $r_P + D = r_I$.

Hence, $D = r_I - r_P = \dfrac{D_I - D_P}{2} = \dfrac{Z_I - Z_P}{2P_d} = \dfrac{m(Z_I - Z_P)}{2}$.

Also, $D = r_s + r_P = \dfrac{D_s + D_P}{2} = \dfrac{Z_s + Z_P}{2P_d} = \dfrac{m(Z_s + Z_P)}{2}$

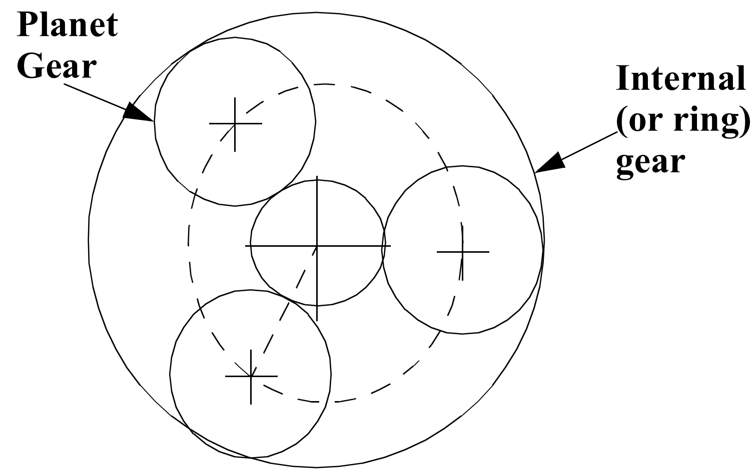
In a planetary gear system, the center distance between the Planet gear (shown) and the sun gear (not shown) follows the same rules as for ordinary spur gears.

This yields the first constraint on planetary gear teeth, $\dfrac{Z_I - Z_P}{2P_d} = \dfrac{Z_s + Z_P}{2P_d}$

$$Z_I = Z_s + 2Z_P$$

Second Law of Planetary Gearing

In order for the planetary gears to be in mesh simultaneously, a restriction on the number of teeth in the ring gear, planetary gears, and sun gear occurs. Although it is not necessary to have the planets in mesh simultaneously, this may result in imbalance in the design and care must also be taken to insure that gear tooth stresses are calculated more carefully.



$$\frac{(Z_s + Z_I)}{N} = \text{an integer},$$

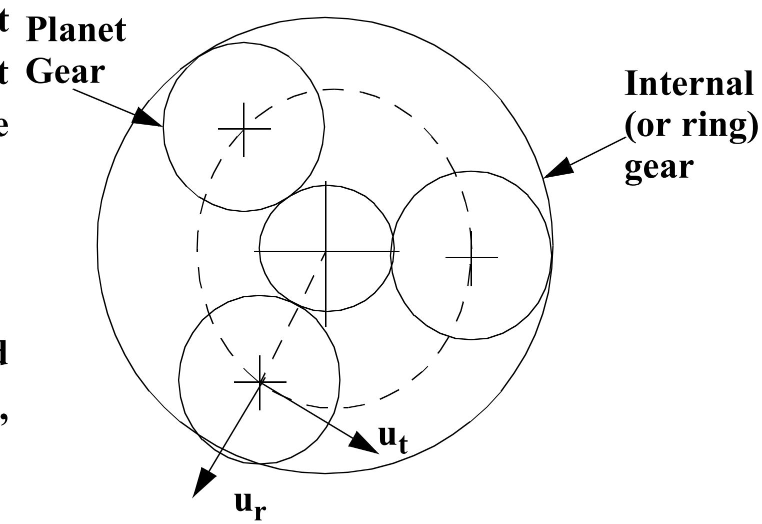
where N is the number of planets (usually 3 or 4).

This condition isn't unbreakable. However, failure to satisfy it will result in the gear teeth not meshing smoothly and correctly and can result in excessive stresses and poor fit.

Note: there is a third law regarding interference among the planet gears themselves. However, if you are pushing that limit, you might want to rethink your overall design. It will be ignored here. Just don't make the planet gears too big.

Speed Ratio in Planetary Gear Systems

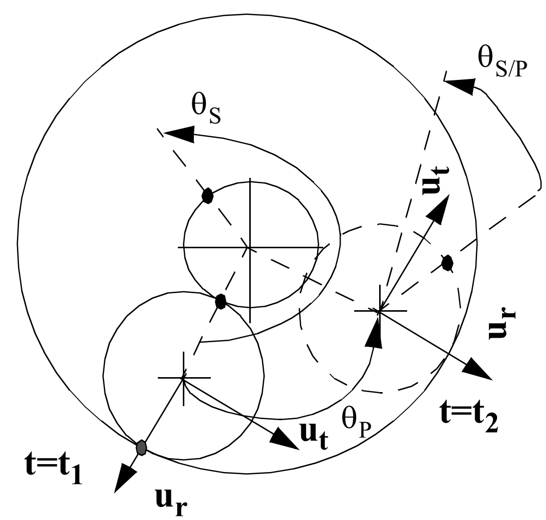
A planetary gear system involves a ring gear (Z_I), several planet gears (Z_p), and a sun gear (Z_s). The sun gear serves as the input element. The planet gears are constrained to move between the sun and the ring gear and serve as the output motion.



The ring gear can be fixed, moving, or driven at a desired speed.

Coordinate System: The $\underset{\sim}{u}_r, \underset{\sim}{u}_t, \underset{\sim}{k}$ coordinate system is body fixed to the planet gear and rotates with the gear with angular velocity, $\dot{\theta}_P \underset{\sim}{k}$.

The sun gear and the ring gear rotate about the fixed center and can be expressed in an inertial coordinate system.

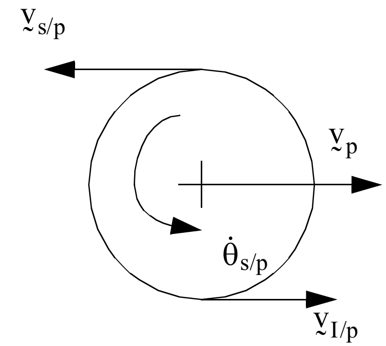


Assuming the sun gear is being driven at a rate $\dot{\theta}_s$ (i.e. counter-clockwise), the pitch velocity is $\underset{\sim}{v}_s = \frac{Z_s \dot{\theta}_s}{2P_d} \hat{u}_t$.

The velocity of the point of contact between the ring gear and the planet gear is

$$\underset{\sim}{v}_I = \frac{Z_I \dot{\theta}_I}{2P_d} \hat{u}_t .$$

Planetary Gear Speed Ratio



Planet Gear

Since the planet gear is both revolving around a center and rotating, determining its surface velocities is not as easy. The position of the center of the planet is $\underset{\sim}{r}_p = \frac{(Z_p + Z_s)}{2P_d}\hat{u}_r$**, where** $\hat{u}_r$ **is the unit vector directed from the center of the sun to the center of the moving planet. This unit vector rotates with angular velocity** $\dot{\theta}_p\hat{k}$**, where** $\hat{k}$ **points out of the page.**

The velocity of the center is $\underset{\sim}{v}_p = \frac{(Z_p + Z_s)}{2P_d}\dot{\theta}_p\hat{u}_t$**, where** $\hat{u}_t$ **is the unit vector in the direction of travel.**

The relative velocity of the point of contact of the planet in contact with the sun is $\underset{\sim}{v}_{s/p} = \left(-\frac{Z_p}{2P_d}\right)\dot{\theta}_{s/p}\hat{u}_t$ **where** $\dot{\theta}_{s/p}$ **is the angular velocity of the planet gear about its axis.**

The relative velocity of the planet point of contact with the ring gear is $\underset{\sim}{v}_{I/p} = \left(\frac{Z_p}{2P_d}\right)\dot{\theta}_{s/p}\hat{u}_t$**.**

The velocity of the point of contact is the same for the sun gear and the planet gear:

$$\underset{\sim}{v}_s = \frac{Z_s\dot{\theta}_s}{2P_d}\hat{u}_t = \frac{(Z_p + Z_s)}{2P_d}\dot{\theta}_p\hat{u}_t - \left(\frac{Z_p}{2P_d}\right)\dot{\theta}_{s/p}\hat{u}_t \quad \textbf{or} \quad Z_s\dot{\theta}_s = (Z_p + Z_s)\dot{\theta}_p - Z_p\dot{\theta}_{s/p} \qquad \textbf{(1)}$$

The velocity of the point of contact is the same for the ring gear and the planet gear:

Planetary Gear Ratio

$$\underset{\sim}{v}_I = \frac{Z_I\dot{\theta}_I}{2P_d}\hat{u}_t = \frac{(Z_p + Z_s)}{2P_d}\dot{\theta}_p\hat{u}_t + \left(\frac{Z_p}{2P_d}\right)\dot{\theta}_{s/p}\hat{u}_t \textbf{ or } Z_I\dot{\theta}_I = (Z_p + Z_s)\dot{\theta}_p + Z_p\dot{\theta}_{s/p} \quad \textbf{(2)}$$

Equation (1) can be solved to determine $\dot{\theta}_{s/p} = \left(1 + \frac{Z_s}{Z_p}\right)\dot{\theta}_p - \left(\frac{Z_s}{Z_p}\right)\dot{\theta}_s$ **,which, when substituted into Equation (2) yields:**

$$\frac{Z_I\dot{\theta}_I + Z_s\dot{\theta}_s}{2(Z_p + Z_s)} = \dot{\theta}_p \quad \textbf{(3)}$$

Since $Z_I = Z_s + 2Z_P$**,**

$$\frac{Z_I\dot{\theta}_I + Z_s\dot{\theta}_s}{Z_I + Z_s} = \dot{\theta}_p \quad \textbf{(4)}$$

The most common situation pins the ring gear, which yields, $\frac{\dot{\theta}_p}{\dot{\theta}_s} = \frac{1}{\frac{Z_I}{Z_s} + 1}$ **.**

If, on the other hand, the ring gear is free to move, it will turn with $\dot{\theta}_I = \dot{\theta}_s$ **and equation (1) becomes:**

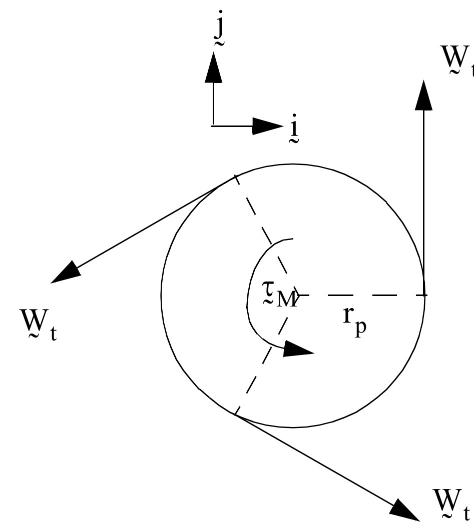
Continuous Variable Transmission

$$\dot{\theta}_p = \frac{Z_I\dot{\theta}_s + Z_s\dot{\theta}_s}{Z_I + Z_s} = \dot{\theta}_s \quad \textbf{(1)}$$

If the internal gear is turned such that $\dot{\theta}_I = k\dot{\theta}_s$**, where** $k \in [0, 1]$**, then**

$$\frac{\dot{\theta}_p}{\dot{\theta}_s} = \frac{kZ_I + Z_s}{Z_I + Z_s} = \frac{k\left(\frac{Z_I}{Z_s}\right) + 1}{\frac{Z_I}{Z_s} + 1} \quad \textbf{or} \quad \frac{\dot{\theta}_p}{\dot{\theta}_s} \in \left[\frac{1}{\frac{Z_I}{Z_s} + 1}, 1\right] \quad \textbf{(2)}$$

Forces in Planetary Systems



In a planetary system, the sun gear is in contact with three or four planet gears.

A free body diagram of the sun gear shows that the tangent forces of contact with the N planetary gears (3 are shown in the FBD), are evenly distributed around the circumference.

By symmetry, the magnitudes of the forces are equal. The moment equation is the only one required to determine the magnitude

Solving yields the following relations:

$$\tau_M = 3r_pW_t$$

Hence, the magnitude of the tangential force is divided by a factor equal to the number of planetary gears.

The net force created by the tangential component of the force is also zero. In other words, it does not provide any load on the bearings.

Stresses in Spur Gears

An estimate for the maximum stress in a gear tooth can be formed by treating the tooth as a rectangular beam of tooth thickness, t, and tooth width, F.

$$\sigma = \frac{6W_tL}{Ft^2} \qquad \textbf{(1)}$$

where σ is the axial stress at the base of the tooth, W_t is the tangential force (computed earlier), and L is the theoretical length of the tooth, where the force acts.

This last assumption is known to be bad. However, considerable literature exists based on this assumption and it gives a more conservative failure estimate than the actual worst case contact point, the mid-point of the tooth.

It is convenient to remove the difficult to handle variables and incorporate a fudge factor which accounts for the tooth's non-rectangular geometry. To this end, the Lewis Form Factor, Y, is created. Tables for this factor for many conditions can be obtained in American Gear Manufacturer Association (AGMA) standards. This information may also be available from gear manufacturers themselves.

Equation (1) can be rewritten as:

$$\sigma = \frac{6W_tY}{FP_d}. \qquad \textbf{(2)}$$

Note: in the modern age, these approximations can be replaced by using a Finite Element Model of the gear tooth.

If you don’t have a Lewis Factor chart

The tooth length is approximately the dedendum plus the addendum, $L = \frac{1.25}{P_d} + \frac{1}{P_d} = \frac{2.25}{P_d} = m(2.25)$**.**

The tooth thickness at the pitch diameter is $t = \frac{\pi}{2P_d} = \frac{\pi m}{2}$**. Note, this is a smaller than the tooth thickness at the root. However, it will predict a more conservative value of the maximum stress and somewhat compensates for the fact that the tooth is not a rectangular beam.**

Using these values in equation (1) on the previous page yields:

$$\sigma = \frac{6W_t\left(\frac{2.25}{P_d}\right)}{F\left(\frac{\pi}{2P_d}\right)^2} = \frac{54P_dW_t}{F\pi^2} \qquad (1)$$

Although this last equation is not standard, it can give an estimate of the actual stress in a gear tooth, lacking more accurate documentation.

The maximum stress in a gear tooth must not exceed the allowable stress, which is determined by the yield stress of the material.

Normally, safety factors are applied based on loading conditions. In the AGMA standard for designing gear teeth, there are about 10 different safety factors.

Worm Gearing

A worm mesh is composed of a <u>worm gear</u> and a <u>worm</u>.

The worm is similar to a rotating rack. It is similar to a helical gear with a very high helix angle. Worm and worm gear systems come in either right hand or left hand varieties. You cannot mix a right hand worm with a left hand worm gear!

Worms usually have one, two, or four threads. The number of <u>thread starts</u> can be seen by looking down the axis of the worm.

For calculating center distances, the pitch diameter of the worm is (P_{dn} is normal diametral pitch, m_n is normal module)

$$d_w = \frac{z_w}{P_{dn}\sin\lambda} = \frac{z_w m_n}{\sin\lambda} \qquad (1)$$

where z_w is the number of thread starts (or teeth) and λ is the lead angle. This angle represents the back-drivability of a worm/gear mesh. The lead for a worm is $L = \frac{\pi z_w}{P_{dn}\cos\lambda}$.

In English gears, the diametral pitch, $P_d = P_{dn}\cos\lambda$, is normally used and $d_w = \frac{z_w}{P_d \tan\lambda}$.

Normally the pitch diameter for both worm and worm gear will be given and calculating it is not necessary.

Worm Gear Quantities

The pitch diameter for the helical worm gear is (P_d is diametral pitch, m_n is normal module)

$$d_g = \frac{z_g}{P_d} = \frac{z_g m_n}{\cos\lambda} \quad \textbf{(1)}$$

The center distance for a worm/gear mesh is:

$$D = \frac{1}{2}(d_w + d_g) = \frac{1}{2P_d}\left(\frac{z_w}{\tan\lambda} + z_g\right) = \frac{m_n}{2}\left(\frac{z_w}{\sin\lambda} + \frac{z_g}{\cos\lambda}\right) \quad \textbf{(2)}$$

Worm/gear meshes can generate tremendous gear ratios in a compact space. They also allow the turning of the drive axis through $90°$.

Because worm/gear meshes involve both rolling contact (to transmit the loads) and sliding contact, they are not as efficient as spur gear systems. Further, since the axis is rotated, thrust loads, which may not be present in planar gear systems, are significant. This increases the requirements on the bearing system in a worm/gear mesh.

The velocity (gear) ratio of a worm/gear mesh is

$$g = \frac{\dot{\theta}_g}{\dot{\theta}_w} = \frac{z_g}{z_w} \quad \textbf{(3)}$$

Self-locking of Worm/Gear mesh

If $\cos\phi_n \sin\lambda - \mu\cos\lambda \le 0$**, the worm/gear mesh will possess self-locking capability.**

The variable, ϕ_n**, is the normal pressure angle (e.g.** $20°$, $25°$, $30°$**). The variable,** μ**, is the coefficient of friction between the worm and the worm gear. This number is usually kept small (especially with lubrication) and varies between values of 0.01 (well lubricated) to 0.20 (poorly lubricated).**

This means that the worm gear cannot drive the worm. Since the normal condition is for the worm to drive the worm gear, self-locking is usually expressed as the inability of the system to be "back-driven."

In the case of drive systems, this is nice, because the motor can drive the wheel, but the wheel cannot drive the motor.

The formula indicates that for self-locking, small lead angle, λ**, must be chosen. The gears in the Stock Drive catalog have lead angles of** $3° - 10°$**.**

Lead angle, $\lambda = \tan{-1}\left(\frac{z_w}{P_d d_w}\right)$**, is directly proportional to the number of teeth. Therefore, fewer teeth results in smaller lead angles, and better self-locking capabilities.**

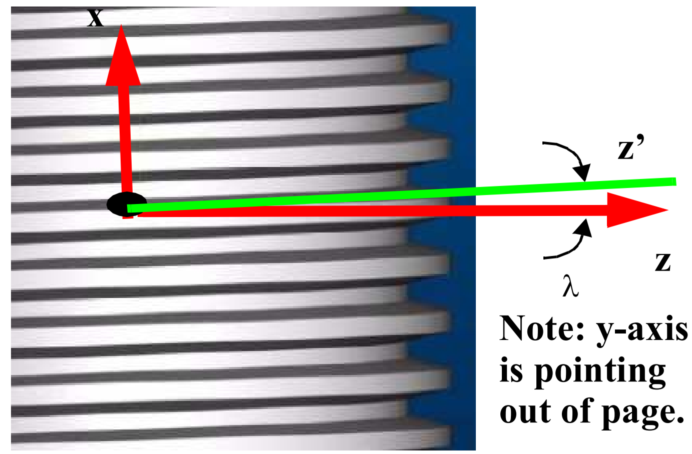
Buford's lead angle, with a four tooth worm, was rather high and did not "self-lock."

We retrofitted using a single tooth worm (same gear separation as with the four tooth gear), and the current system, while slower, does self-lock.

Coordinate Systems in Worm

The worm has a "rack-type" tooth form and resembles a helical gear with a high helix angle.

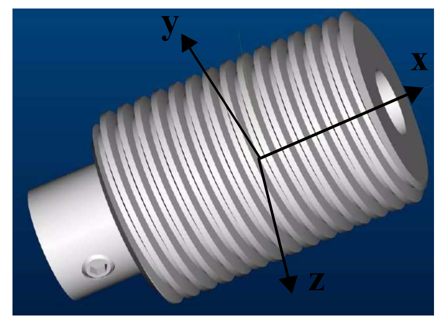
Looking down on the worm, the helix angle at the contact point is defined as seen in the figure, and is called the lead angle, λ.



Note: y-axis is pointing out of page.

Consider two Cartesian coordinate systems. The first has its x axis aligned with the axis of the worm, its y-axis aligned along the radius of the worm, and its z-axis perpendicular to the x and y axes in a right handed fashion.

The second coordinate system is rotated about the y axis through the lead angle. A coordinate transformation matrix for this transformation is $\begin{bmatrix} \cos\lambda & 0 & -\sin\lambda \\ 0 & 1 & 0 \\ \sin\lambda & 0 & \cos\lambda \end{bmatrix}$.

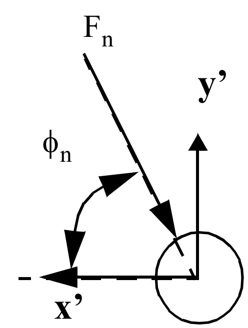


Right-handed worm

Force Definitions in Worm

In a worm tooth, there is a normal force, F_n, which is directed perpendicular to the tooth at the pitch circle and which is directed along the normal pressure angle, ϕ_n.

There is also a friction force, F_f, which is directed along the tooth and always opposes the motion. If sliding is occuring, then this force is $F_f = \mu_k F_n$. If self-locking is occuring, then this force has the magnitude, $|F_f| \le \mu_s F_n$.



The vector force, in the transformed coordinate system, acting on the worm tooth is:

$$\underset{\sim}{F}_w = F_n(-\cos(\phi_n)\underset{\sim}{u}_{x'} - \sin(\phi_n)\underset{\sim}{u}_{y'}) + (F_f)\underset{\sim}{u}_{z'}$$

Since the positive rotation of the worm is counterclockwise, the friction force acts along the positive z' axis when opposing the motion.

Transforming this back into the original coordinate system gives:

$$\begin{bmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{bmatrix} \begin{bmatrix} -F_n\cos(\phi_n) \\ -F_n\sin(\phi_n) \\ F_f \end{bmatrix} = \begin{bmatrix} -F_n\cos(\phi_n)\cos\lambda + F_f\sin\lambda \\ -F_n\sin(\phi_n) \\ F_n\cos(\phi_n)\sin\lambda + F_f\cos\lambda \end{bmatrix} = \begin{bmatrix} F_{a1} \\ F_{r1} \\ F_{u1} \end{bmatrix} = \begin{bmatrix} -F_{u2} \\ -F_{r2} \\ -F_{a2} \end{bmatrix}.$$

The transpose of the CTM was used to convert from transformed coordinates back into original coordinates.

Forces is Worm/Gear Meshes

The driving torque is $\underset{\sim}{\tau}_M = \tau_M \underset{\sim}{u}_x$. The worm forces act at the radius, $\underset{\sim}{r} = \left(\frac{d_w}{2}\right)\underset{\sim}{u}_y$. The x component of the moment equation yields

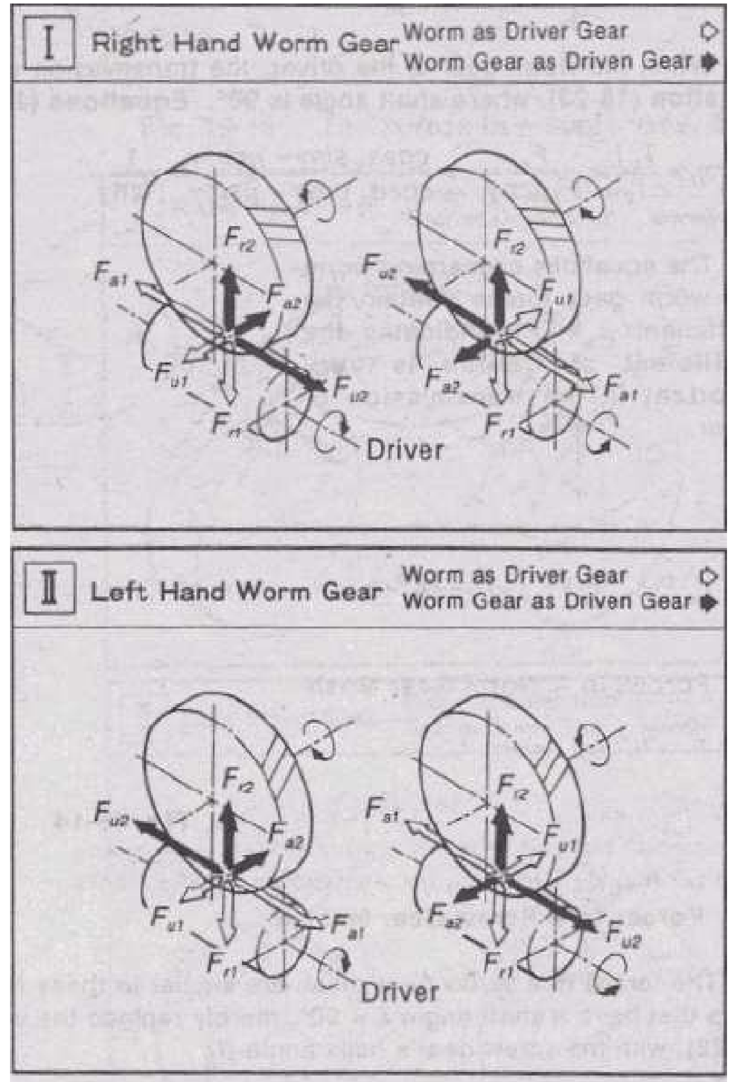
$$\tau_M = -\left(\frac{d_w}{2}\right)(F_n\cos(\phi_n)\sin\lambda + F_f\cos\lambda) \ .$$

The driven torque is $\underset{\sim}{\tau}_D = \tau_D \underset{\sim}{u}_z$. The worm gear forces act at the radius, $\underset{\sim}{r} = -\left(\frac{d_g}{2}\right)\underset{\sim}{u}_y$. The z component of the moment equation yields $\tau_D = \left(\frac{d_g}{2}\right)(-F_n\cos(\phi_n)\cos\lambda + F_f\sin\lambda)$.

The torque efficiency is the ratio of these two torques:

$$\frac{\tau_D}{\tau_M} = \left(\frac{d_g}{d_w}\right)\frac{F_n\cos(\phi_n)\cos\lambda - F_f\sin\lambda}{F_n\cos(\phi_n)\sin\lambda + F_f\cos\lambda}.$$

For sliding, this becomes: $\frac{\tau_D}{\tau_M} = \left(\frac{d_g}{d_w}\right)\frac{\cos(\phi_n)\cos\lambda - \mu_k\sin\lambda}{\cos(\phi_n)\sin\lambda + \mu_k\cos\lambda}$



Worm and Gear
(reprinted from Stock Drive Design Guide)

Driving Forces of Worm Gears

For determining the worm/gear mesh efficiency, the bearing loads can be estimated as the usually 90% loss (ie f=.9). The coefficient of friction can be estimated, and the loss due to tooth-tooth sliding can be computed directly.

Using the definitions of pitch diameter and gear ratio,

$$\frac{d_g}{d_w} = \frac{\dfrac{z_g p_n}{\pi \cos\lambda}}{\dfrac{z_w p_n}{\pi \sin\lambda}} = \frac{z_g \tan\lambda}{z_w} = g \tan\lambda . \qquad \textbf{(1)}$$

Thus,

$$\frac{\tau_D}{\tau_M} = (fg)\frac{\cos(\phi_n)\cos\lambda - \mu_k \sin\lambda}{\cos(\phi_n)\sin\lambda + \mu_k \cos\lambda}\tan\lambda . \qquad \textbf{(2)}$$

Normal Force

The normal force can be calculated in terms of the driving torque, $\tau_M = \left(\frac{d_w}{2}\right)F_{u1}$**.**

Since $F_{u1} = (\cos\phi_n \sin\lambda + \mu\cos\lambda)F_n$**,** $F_n = \frac{2\tau_M}{(\cos\phi_n \sin\lambda + \mu\cos\lambda)d_w}$**.**

Using the expressions for pitch diameter, $d_w = \frac{z_w}{P_d \tan\lambda}$ **or** $d_w = \frac{z_w m_n}{\sin\lambda}$**, the normal force is:**

$$F_n = \frac{2\tau_M(P_d \tan\lambda)}{(\cos\phi_n \sin\lambda + \mu\cos\lambda)z_w} \qquad \textbf{(1)}$$

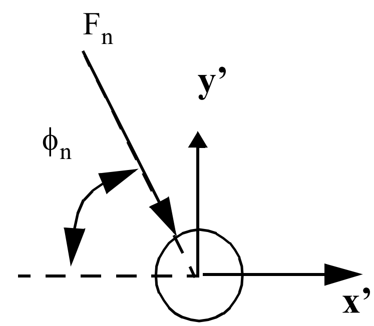
or

$$F_n = \frac{2\tau_M \sin\lambda}{(\cos\phi_n \sin\lambda + \mu\cos\lambda)(z_w m_n)}. \qquad \textbf{(2)}$$

This force is used in calculating failure of gear teeth.

Back-Driving Worm and Gear

When the worm gear is driving, the normal force acts on the opposite tooth face as when the worm is driving.



$$\underset{\sim}{F}_w = F_n(\cos(\phi_n)\underset{\sim}{u}_{x'} - \sin(\phi_n)\underset{\sim}{u}_{y'}) + (F_f)\underset{\sim}{u}_{z'} \ .$$

$$\begin{bmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{bmatrix} \begin{bmatrix} F_n\cos(\phi_n) \\ -F_n\sin(\phi_n) \\ F_f \end{bmatrix} = \begin{bmatrix} F_n\cos(\phi_n)\cos\lambda + F_f\sin\lambda \\ -F_n\sin(\phi_n) \\ -F_n\cos(\phi_n)\sin\lambda + F_f\cos\lambda \end{bmatrix} = \begin{bmatrix} F_{a1} \\ F_{r1} \\ F_{u1} \end{bmatrix} = \begin{bmatrix} -F_{u2} \\ -F_{r2} \\ -F_{a2} \end{bmatrix}$$

$$\tau_M = -\left(\frac{d_w}{2}\right)(-F_n\cos(\phi_n)\sin\lambda + F_f\cos\lambda) \ \textbf{and} \ \tau_D = \left(\frac{d_g}{2}\right)(F_n\cos(\phi_n)\cos\lambda + F_f\sin\lambda) .$$

The ratio of drive torque to driving torque is:

$$\frac{\tau_D}{\tau_M} = \frac{d_g}{d_w}\frac{F_n\cos(\phi_n)\cos\lambda + F_f\sin\lambda}{F_n\cos(\phi_n)\sin\lambda - F_f\cos\lambda}$$

In the event of self-locking, the denominator would be zero and the static friction coefficient would apply:

$$\cos(\phi_n)\sin\lambda - \mu_s\cos\lambda = 0 .$$

Example: Worm/Gear Mesh

Consider a left handed worm/gear from the Stock Drive Catalog which has the following parameters:

General: pressure angle (ϕ_n**) =** $25°$**, pitch (**$P_d = 24$**), lead angle (**$\lambda = 18.43°$**), maximum drive torque:** $\tau_D = 10\text{inlb}$

Assume a coefficient of sliding friction of 0.05.

Worm: 4 threads **Worm Gear: 20 teeth**

Velocity ratio: $g = \frac{20}{4} = 5$**, which hardly seems worth it.**

Drive ratio: $\frac{\tau_D}{\tau_M} = (fg)\frac{(\cos\phi_n\cos\lambda - \mu\sin\lambda)}{(\cos\phi_n\sin\lambda + \mu\cos\lambda)}\tan\lambda = (0.9)(5)\frac{\cos(25°)\cos(18.43°) - 0.05\sin(18.43°)}{\cos(25°)\sin(18.43°) + 0.05\cos(18.43°)}\tan(18.43°) = 3.8$**.**

Normal force: $F_n = \frac{2\tau_M(P_d\tan\lambda)}{(\cos\phi_n\sin\lambda + \mu\cos\lambda)z_w} = \frac{2(10\text{inlb})(24\text{in}^{-1})\tan(18.43°)}{(\cos(25°)\sin(18.43°) + (0.05)\cos(18.43°))(4)} = 120\text{lb}$

Worm Axial load: $F_{a1} = (\cos\phi_n\cos\lambda - \mu\sin\lambda)F_n = (\cos(25°)\cos(18.43°) - 0.05\sin(18.43°))120\text{lb} = 100\text{lb}$**.**

Tangential force: $F_1 = F_n\cos\phi_n = (120\text{lb})\cos(25°) = 110\text{lb}$

Worm Gear Axial load: $F_{a2} = (\cos\phi_n\sin\lambda + \mu\cos\lambda)F_n = (\cos(25°)\sin(18.43°) + 0.05\cos(18.43°))120\text{lb} = 40\text{lb}$**.**