Mathematical Preliminaries

Orthogonal Coordinates: Let the vector, $\underset{\sim}{v} = x_1\underset{\sim}{e}_1 + x_2\underset{\sim}{e}_2 + x_3\underset{\sim}{e}_3$**, be expressed in a right-handed coordinate system with components,** (x_1, x_2, x_3) **, of the unit vectors,** $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ **. The unit vectors form a <u>basis</u> for the system. The components are scalars (i.e. real numbers).**

Use the compact indicial notation to represent the set of three coordinates by $\underset{\sim}{e}_i$ **for** $i = 1, 2, 3$ **.**

Since $\underset{\sim}{e}_i$ **are unit vectors,** $\|\underset{\sim}{e}_i\| = \left[\sum_{i=1}^{3} \underset{\sim}{e}_i \bullet \underset{\sim}{e}_i\right]^{\frac{1}{2}} = 1$ **.**

Repeated indices imply summation. Therefore, the vector can be represented compactly as $\underset{\sim}{v} = \sum_{i=1}^{3} x_i\underset{\sim}{e}_i = x_i\underset{\sim}{e}_i$**.**

Since the summation convention applies to repeated indices, an index cannot be used more than once in an expression. For complicated expressions, this is usually a good test to determine if errors have been made.

The vector, $\underset{\sim}{v}$**, is an <u>invariant</u>. Its components,** x_i**, will vary with coordinate system, but the vector itself <u>does not change</u>. Hence,** $\underset{\sim}{v} = x_i\underset{\sim}{e}_i = x_i'\underset{\sim}{e}_i'$ **.**

Coordinate Transformation Laws

Inner (Dot) Product: The inner product between two invariants is denoted by:

$$\underset{\sim}{v} \bullet \underset{\sim}{w} = \sum_{i=1}^{3}(v_i \underset{\sim}{e}_i) \bullet \sum_{j=1}^{3}(w_j \underset{\sim}{e}_j) = \sum_{i=1}^{3}\sum_{j=1}^{3}(v_i w_j)(\underset{\sim}{e}_i \bullet \underset{\sim}{e}_j).$$

The values for the Inner Product between two unit vectors in the same coordinate system is $\underset{\sim}{e}_i \bullet \underset{\sim}{e}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} = \delta_{ij}$.

The quantity, δ_{ij}**, is called the Kronecker delta.**

The values for the Inner Product between two unit vectors in the different coordinate systems is $\underset{\sim}{e}_{i'} \bullet \underset{\sim}{e}_j = \cos\theta_{i'j} = a_{i'j}$ **, where** $\theta_{i'j}$ **is the inner angle between the two unit vectors.**

Note that $a_{i'j}$ **is <u>not symmetric</u>!**

The dot product can be reduced using the Kronecker delta:

$$\underset{\sim}{v} \bullet \underset{\sim}{w} = \sum_{i=1}^{3}\sum_{j=1}^{3}(v_i w_j)(\underset{\sim}{e}_i \bullet \underset{\sim}{e}_j) = \sum_{i=1}^{3}\sum_{j=1}^{3} v_i w_j \delta_{ij} = \sum_{i=1}^{3} v_i(\delta_{i1}w_1 + \delta_{i2}w_2 + \delta_{i3}w_3) = ,$$

$$v_1(\delta_{11}w_1 + \delta_{12}w_2 + \delta_{33}w_3) + v_2(\delta_{21}w_1 + \delta_{22}w_2 + \delta_{23}w_3) + v_3(\delta_{31}w_1 + \delta_{32}w_2 + \delta_{33}w_3) = v_1w_1 + v_2w_2 + v_3w_3 = v_iw_i$$

where the summation has been suppressed on the final expression.

Inverse of the CTM

Consider a vector expressed in two different orthogonal coordinate systems: $\underset{\sim}{v} = x_i \underset{\sim}{e}_i = x_{i'}' \underset{\sim}{e}_{i'}'$

Form the dot product for this equation with the unit vectors from the transformed system, $\underset{\sim}{e}_{k'}'$ **, to <u>contract</u> it:**

$$\underset{\sim}{v} \bullet \underset{\sim}{e}_{k'}' = (x_i \underset{\sim}{e}_i) \bullet \underset{\sim}{e}_{k'}' = (x_{i'}' \underset{\sim}{e}_{i'}') \bullet \underset{\sim}{e}_{k'}'$$

This reduces to: $x_i(\underset{\sim}{e}_i \bullet \underset{\sim}{e}_{k'}') = x_{i'}'(\underset{\sim}{e}_{i'}' \bullet \underset{\sim}{e}_{k'}')$.

Replacing the dot products yields: $x_i a_{k'i} = x_{i'}'\delta_{k'i'} = x_1'\delta_{k'1} + x_2'\delta_{k'2} + x_3'\delta_{k'3}$.

The right hand side can be further reduced by noting that :

$$\begin{aligned} k' &= 1,\ x_1'\delta_{11} + x_2'\delta_{12} + x_3'\delta_{13} = x_1' \\ k' &= 2,\ x_1'\delta_{21} + x_2'\delta_{22} + x_3'\delta_{23} = x_2' \\ k' &= 3,\ x_1'\delta_{31} + x_2'\delta_{32} + x_3'\delta_{33} = x_3' \end{aligned} \quad .$$

Therefore, the coordinate transformation law for a vector is seen to be: $x_i a_{k'i} = x_{k'}'$

An important property of the coordinate transformation matrix, $a_{i'j}$ **, can be seen by transforming a unit vector,**

$$\underset{\sim}{v} = \delta_{i'1} \underset{\sim}{e}_{i'}' = \underset{\sim}{e}_1' = x_i \underset{\sim}{e}_i \quad . \qquad \textbf{(1)}$$

Inverse of the CTM

The components, x_i**, can be determined by contracting** $\underset{\sim}{v}$**:**

$$\underset{\sim}{v} \bullet \underset{\sim}{e}_i = \underset{\sim}{e}_1' \bullet \underset{\sim}{e}_i = x_i \quad .$$

By the definition of the CTM, $x_i = \underset{\sim}{e}_1' \bullet \underset{\sim}{e}_i = a_{1i}$ **. Substituting** $x_i = a_{1i}$ **into Equation (1) on Page 3 yields** $\underset{\sim}{v} = a_{1i}\underset{\sim}{e}_i$**.**

Contracting by $\underset{\sim}{e}_k'$ **yields:** $\underset{\sim}{e}_1' \bullet \underset{\sim}{e}_{k'}' = a_{1i}\underset{\sim}{e}_i \bullet \underset{\sim}{e}_{k'}' = a_{1i}a_{k'i} = a_{1i}a^T_{ik'} = \delta_{1k'}$ **.**

Repeating this procedure for the other two unit vectors, $\underset{\sim}{e}_2'$ **and** $\underset{\sim}{e}_3'$**, yields and** $a_{2i}a^T_{ik'} = \delta_{2k'}$ **and** $a_{3i}a^T_{ik'} = \delta_{3k'}$ **.**

Hence, the coordinate transformation matrix, CTM, is orthogonal: $\delta_{j'k'} = a_{j'i}a^T_{ik'}$ **, and the inverse is the transpose.**

This can be expressed in matrix notation as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad .$$

Properties of Vectors

Addition of Vectors: $\underset{\sim}{A} + \underset{\sim}{B} = \underset{\sim}{C}$**. Hence,** $A_i\underset{\sim}{e}_i + B_i\underset{\sim}{e}_i = C_i\underset{\sim}{e}_i$**, or** $A_i + B_i = C_i$**.**

Inner (Scalar) Product: $\underset{\sim}{A} \bullet \underset{\sim}{B} = (A_i\underset{\sim}{e}_i) \bullet (B_j\underset{\sim}{e}_j) = A_iB_j(\underset{\sim}{e}_i \bullet \underset{\sim}{e}_j) = A_iB_j\delta_{ij} = A_iB_i = \|\underset{\sim}{A}\|\|\underset{\sim}{B}\|\cos\theta$

Outer Product: $\underset{\sim}{A}\underset{\sim}{B} = (A_i\underset{\sim}{e}_i)(B_j\underset{\sim}{e}_j) = A_iB_j(\underset{\sim}{e}_i\underset{\sim}{e}_j)$

Dyads: The quantity, $\underset{\sim}{e}_i\underset{\sim}{e}_j$**, is called a dyad or dyadic. Order in the dyad is important:** $\underset{\sim}{e}_i\underset{\sim}{e}_j \neq \underset{\sim}{e}_j\underset{\sim}{e}_i$ **.**

However, $(a\underset{\sim}{e}_i)\underset{\sim}{e}_j = a(\underset{\sim}{e}_i\underset{\sim}{e}_j) = \underset{\sim}{e}_i(a\underset{\sim}{e}_j)$ **.**

Let $P_{ij} = A_iB_j = \begin{bmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{bmatrix}$**. Then, the invariant,** $\underset{\sim}{P} = A_iB_j(\underset{\sim}{e}_i\underset{\sim}{e}_j) = P_{ij}(\underset{\sim}{e}_i\underset{\sim}{e}_j)$ **.**

Since $\underset{\sim}{P}$ **is an invariant,** $\underset{\sim}{P} = \underset{\sim}{e}_iP_{ij}\underset{\sim}{e}_j = \underset{\sim}{e}'_{i'}P'_{i'j'}\underset{\sim}{e}'_{j'}$**.**

Contracting on $\underset{\sim}{e}'_{k'}$**, yields:** $\underset{\sim}{P} \bullet \underset{\sim}{e}'_{k'} = \underset{\sim}{e}_iP_{ij}(\underset{\sim}{e}_j \bullet \underset{\sim}{e}'_{k'}) = \underset{\sim}{e}'_{i'}P'_{i'j'}(\underset{\sim}{e}'_{j'} \bullet \underset{\sim}{e}'_{k'})$ **, which reduces to** $\underset{\sim}{e}_iP_{ij}a_{k'j} = \underset{\sim}{e}'_{i'}P'_{i'k'}$**.**

Contracting on $\underset{\sim}{e}'_{l'}$**, yields:** $\underset{\sim}{e}'_{l'} \bullet \underset{\sim}{P} \bullet \underset{\sim}{e}'_{k'} = a_{l'i}P_{ij}a_{k'j} = P'_{l'k'}$**, which is the transformation law for a second rank tensor.**

Similarly, $\underset{\sim}{e}_i\underset{\sim}{e}_j\underset{\sim}{e}_k$ **is a tryadic. Hence,** $a_{l'i}a_{m'j}a_{n'k}Q_{ijk} = Q'_{l'm'n'}$**.**

Cross Product: $\underset{\sim}{A} \times \underset{\sim}{B} = e_{ijk}A_iB_j\underset{\sim}{e}_k$**, where** e_{ijk} **is the cyclic permutation tensor.**

Position, Velocity, and Acceleration of a Point

Kinematics = study of the motions of particles and rigid bodies, disregarding the forces associated with these motions.

Reference frames = Set of coordinates in which position, velocity, acceleration are specified.

Inertial Reference frame = a reference frame whose origin is either

a) fixed in space and time or

b) translating uniformly and not rotating.

Relative Reference frame = a reference frame which is rotating with respect to an inertial reference frame or whose origin is accelerating relative to an inertial reference frame.

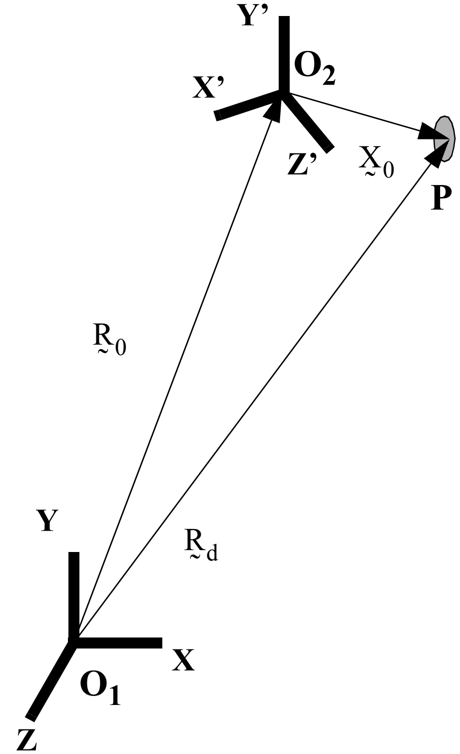
The position of a particle relative to a reference frame is given by the vector, $\underset{\sim}{r}$, drawn from the origin of the frame to the particle.

Velocity, V, of the particle is the total time derivative of the position, $\frac{d\underset{\sim}{r}}{dt}$.

Acceleration, A, of the particle is the total time derivative of the velocity, $\frac{d^2\underset{\sim}{r}}{dt^2}$

Velocity and Acceleration of Rotating Reference Frames

Consider two different means of expressing the position of a point: the first in an inertial reference frame ($\underset{\sim}{E}_I$), the second in a rotating reference frame ($\underset{\sim}{e}_i$), which is defined such that the components of the vector, $\underset{\sim}{X}_0$, are not time dependent.



There are two expressions for P, $\underset{\sim}{r} = \underset{\sim}{R}_d = R_{dI}\underset{\sim}{E}_I$ **and** $\underset{\sim}{r} = \underset{\sim}{R}_0 + \underset{\sim}{X}_0 = R_{0I}\underset{\sim}{E}_I + X_{0i}\underset{\sim}{e}_i$.

The rotating reference frame has been chosen so that the point, P, appears to be motionless with respect to origin, O_2.

The velocity of P can be determined from the two position expressions:

$$\underset{\sim}{v} = \frac{d}{dt}(R_{dI})\underset{\sim}{E}_I \text{ and } \underset{\sim}{v} = \frac{dR_{0I}}{dt}\underset{\sim}{E}_I + R_{0I}\frac{d(\underset{\sim}{E}_I)}{dt} + \frac{dX_{0i}}{dt}\underset{\sim}{e}_i + X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt} = \frac{dR_{0I}}{dt}\underset{\sim}{E}_I + X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt}.$$

The first term on the RHS, $\frac{dR_{0I}}{dt}\underset{\sim}{E}_I$, is the <u>absolute</u> velocity of the origin, O_2. The second term, $X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt}$, is the <u>relative</u> velocity of P with respect to O_2. The derivative, $\frac{dR_{0I}}{dt}$, is straight-forward to compute, since it is a time derivative of the scalar components, R_{0I}. The derivative, $\frac{d(\underset{\sim}{e}_i)}{dt}$, is a derivative of a vector, and the formulae from basic calculus do not apply.

Angular Velocity

Since $\underset{\sim}{e}_i \bullet \underset{\sim}{E}_I = a_{iI}$**,** $\underset{\sim}{e}_i = a_{iI}\underset{\sim}{E}_I$**. Then,** $\frac{d(\underset{\sim}{e}_i)}{dt} = \frac{da_{iI}}{dt}\underset{\sim}{E}_I + a_{iI}\frac{d(\underset{\sim}{E}_I)}{dt} = \frac{da_{iI}}{dt}\underset{\sim}{E}_I$ **.**

Since $\underset{\sim}{e}_i = a_{iI}\underset{\sim}{E}_I$ **,** $\underset{\sim}{E}_J = a^T_{Ji}\underset{\sim}{e}_i$**. Then,** $\frac{d(\underset{\sim}{e}_j)}{dt} = \frac{d(a_{jI})}{dt}a^T_{Ii}\underset{\sim}{e}_i$ **. Defining** $G_{ji} = \frac{d(a_{jI})}{dt}a^T_{Ii}$ **,**

$$\frac{d(\underset{\sim}{e}_j)}{dt} = G_{ji}\underset{\sim}{e}_i . \qquad \textbf{(1)}$$

Since $\delta_{ji} = a_{jI}a^T_{Ii}$**,** $\frac{d\delta_{ji}}{dt} = 0 = \frac{d(a_{jI}a^T_{Ii})}{dt} = \frac{d(a_{jI})}{dt}a^T_{Ii} + a_{jI}\frac{d(a^T_{Ii})}{dt} = \frac{d(a_{jI})}{dt}a^T_{Ii} + \frac{d(a_{iI})}{dt}a^T_{Ij} = G_{ji} + G_{ij} = G_{ji} + G^T_{ji}$ **.**

Therefore, $G_{ji} = -G^T_{ji}$**. This means that** G_{ji} **is a second rank, anti-symmetric tensor, and only three components are independent:** $G_{ji} = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix}$**. The components,** Ω_i**, are scalars.**

An <u>axial</u> vector can be defined: $\underset{\sim}{\Omega} = \Omega_i\underset{\sim}{e}_i = \frac{1}{2}e_{ijk}G_{jk}\underset{\sim}{e}_i$ **. Using the identity,** $e_{ijk}e_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ **,**

$$e_{ilm}\Omega_i = \frac{1}{2}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})G_{jk} = \frac{1}{2}(G_{lm} - G_{ml}) = \frac{1}{2}(G_{lm} - G^T_{lm}) = G_{lm} . \qquad \textbf{(2)}$$

Definition of Angular Velocity

From Equation (1) on Page 8, the invariant, $\underset{\sim}{\Omega} = \frac{1}{2}e_{ijk}\frac{d(a_{jI})}{dt}a^T_{Ik}\underset{\sim}{e}_i$ **, is the angular velocity of the of the rotating system with respect to the fixed system. Hence it is the rotating system's absolute angular velocity.**

Substituting Equation (2) on Page 8 (with appropriate index substitutions) into Equation (1) on Page 8 yields the derivative of the rotating base vectors in terms of the absolute angular velocity components:

$$\frac{d(\underset{\sim}{e}_i)}{dt} = e_{kij}\Omega_k\underset{\sim}{e}_j \quad . \tag{1}$$

The relative velocity term, $X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt}$**, can be reduced in complexity:** $X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt} = X_{0i}e_{kij}\Omega_k\underset{\sim}{e}_j = e_{kij}\Omega_k X_{0i}\underset{\sim}{e}_j$.

By definition of the cross product in indicial notation (page 5),

$$X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt} = \underset{\sim}{\Omega} \times \underset{\sim}{X}_0 \quad . \tag{2}$$

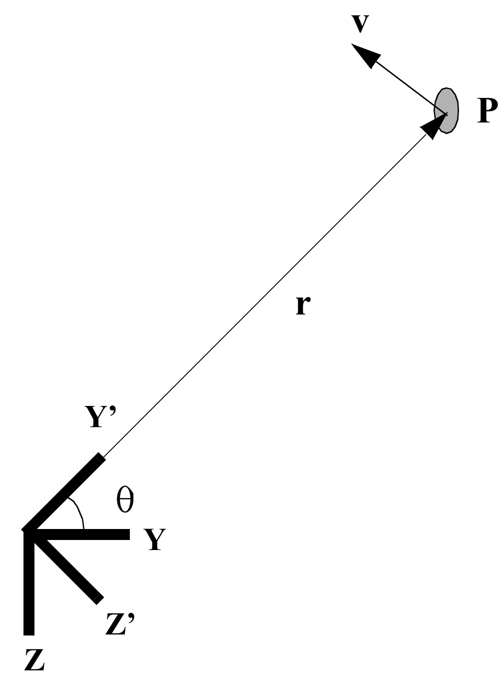
Hence, $\underset{\sim}{v} = \underset{\sim}{v}_{O_2} + \underset{\sim}{\Omega} \times \underset{\sim}{X}_0$.

<u>The absolute velocity of a point, expressed in a rotating coordinate system, is the absolute velocity of the origin of that coordinate system plus the cross product between the vector and the absolute angular velocity of the system.</u>

Example: Circular Motion

Consider an object moving in a circular orbit with radius, r, around a fixed point.

Consider two coordinate systems, one (Y,Z) fixed at the origin of the motion and the second (Y’,Z’) with coincident origin, but which rotates with the object. Note that the X and X’ axes are <u>into</u> the page. The unit vectors for the (X,Y,Z) system are $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ **and for the (X’,Y’,Z’) system are** $(\underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3')$.



The CTM is: $a_{i'j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ **. The antisymmetric matrix is:**

$$G_{j'i'} = \frac{d}{dt}(a_{j'k})a_{ki'}^{T} = \dot{\theta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \dot{\theta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence, the angular velocity is: $\Omega_1 = -\dot{\theta}$ $\quad \Omega_2 = \Omega_3 = 0$ **or** $\underset{\sim}{\Omega} = -\dot{\theta}\underset{\sim}{e}_1'$ **. In other words, the angular velocity for the primed system moving counterclockwise is a vector proportional to the rate of rotation pointing <u>out</u> of the page.**

The position of P is: $\underset{\sim}{r}_P = r\underset{\sim}{e}_2'$**. The velocity of P is** $\underset{\sim}{v}_P = \underset{\sim}{\Omega} \times \underset{\sim}{r}_P = -\dot{\theta}\underset{\sim}{e}_1' \times r\underset{\sim}{e}_2' = -r\dot{\theta}\underset{\sim}{e}_3'$

Relative Motion Inside a Rotating Reference Frame

Returning to the expression for P, $\underset{\sim}{r} = \underset{\sim}{R}_0 + \underset{\sim}{X}_0 = R_{0I}\underset{\sim}{E}_I + X_{0i}\underset{\sim}{e}_i$.

Choose the rotating reference frame so that the point, P, moves with respect to origin, O_2. In this case, the components of the vectors, as seen in the rotating reference frame, are also moving. That is, $X_{0i} = X_{0i}(t)$.

The velocity of P becomes:

$$\underset{\sim}{v} = \frac{d(R_{0I})}{dt}\underset{\sim}{E}_I + R_{0I}\frac{d(\underset{\sim}{E}_I)}{dt} + \frac{d(X_{0i})}{dt}\underset{\sim}{e}_i + X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt} = \frac{d(R_{0I})}{dt}\underset{\sim}{E}_I + [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0] + \frac{d(X_{0i})}{dt}\underset{\sim}{e}_i. \quad \textbf{(1)}$$

This new term, $\frac{d(X_{0i})}{dt}\underset{\sim}{e}_i = \underset{\sim}{v}_{rot}$, **is the velocity of P <u>as seen in the rotating reference frame</u>.**

The term, $\underset{\sim}{v}_{rot}$, **is an invariant vector. However, it is not a true velocity term and is not integrable to a position vector.**

The general equation for the velocity of a vector described in a rotating reference frame is:

$$\underset{\sim}{v} = \frac{d(R_{0I})}{dt}\underset{\sim}{E}_I + [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0] + \underset{\sim}{v}_{rot} \quad . \quad \textbf{(2)}$$

Acceleration

Consider now the acceleration of P:

$$\underset{\sim}{a} = \frac{d(\underset{\sim}{v})}{dt} = \frac{d}{dt}\left\{ \frac{d(R_{0I})}{dt}\underset{\sim}{E}_I + [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0] + \underset{\sim}{v}_{rot} \right\} = \frac{d^2(R_{0I})}{dt^2}\underset{\sim}{E}_I + \frac{d(\underset{\sim}{\Omega} \times \underset{\sim}{X}_0)}{dt} + \frac{d(\underset{\sim}{v}_{rot})}{dt} \quad . \qquad \textbf{(3)}$$

The first term, $\frac{d^2(R_{0I})}{dt^2}\underset{\sim}{E}_I$**, is the absolute acceleration of the origin of the rotating reference from:** $\underset{\sim}{a}_{O_2} = \frac{d^2(R_{0I})}{dt^2}\underset{\sim}{E}_I$**.**

Since $\underset{\sim}{v}_{rot} = \frac{d(X_{0i})}{dt}\underset{\sim}{e}_i = v_{0i}\underset{\sim}{e}_i$**, it is a vector expressed in rotating coordinates with time varying components.**

$$\frac{d(\underset{\sim}{v}_{rot})}{dt} = \frac{d}{dt}\left[\frac{d(X_{0i})}{dt}\underset{\sim}{e}_i\right] = \frac{d^2(X_{0i})}{dt^2}\underset{\sim}{e}_i + \frac{d(X_{0i})}{dt}\frac{d(\underset{\sim}{e}_i)}{dt} = \underset{\sim}{a}_{rot} + \underset{\sim}{\Omega} \times \underset{\sim}{v}_{rot} \quad , \qquad \textbf{(4)}$$

where $\underset{\sim}{a}_{rot}$ **is defined as** $\underset{\sim}{a}_{rot} = \frac{d^2(X_{0i})}{dt^2}\underset{\sim}{e}_i$**.** $\underset{\sim}{a}_{rot}$ **is the <u>acceleration of P as seen in the rotating frame.</u>**

The substitution of $\underset{\sim}{\Omega} \times \underset{\sim}{v}_{rot} = \frac{d(X_{0i})}{dt}\frac{d(\underset{\sim}{e}_i)}{dt}$ **follows directly from Equation (2) on Page 9, where the** $v_{0i} = \frac{d(X_{0i})}{dt}$ **are no different from the** X_{0i}**. That is, time varying components of a vector represented in rotating coordinates.**

Coriolis Acceleration

The second term in Equation (4) on Page 12 becomes: $\frac{d(\underset{\sim}{\Omega} \times \underset{\sim}{X}_0)}{dt} = \frac{d(\underset{\sim}{\Omega})}{dt} \times \underset{\sim}{X}_0 + \underset{\sim}{\Omega} \times \frac{d(\underset{\sim}{X}_0)}{dt}$.

The derivative of the angular velocity is the angular acceleration, $\underset{\sim}{\alpha}$ **and is:**

$$\frac{d(\underset{\sim}{\Omega})}{dt} = \frac{d(\Omega_i \underset{\sim}{e}_i)}{dt} = \frac{d(\Omega_i)}{dt}\underset{\sim}{e}_i + \underset{\sim}{\Omega} \times \underset{\sim}{\Omega} = \frac{d(\Omega_i)}{dt}\underset{\sim}{e}_i = \alpha_i \underset{\sim}{e}_i = \underset{\sim}{\alpha}. \quad \textbf{(5)}$$

The final term becomes: $\underset{\sim}{\Omega} \times \frac{d(\underset{\sim}{X}_0)}{dt} = \underset{\sim}{\Omega} \times \left[\frac{d(X_{0i})}{dt}\underset{\sim}{e}_i + \underset{\sim}{\Omega} \times \underset{\sim}{X}_0\right] = \underset{\sim}{\Omega} \times \underset{\sim}{v}_{rot} + \underset{\sim}{\Omega} \times [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0]$.

Collecting all of the terms and substituting into Equation (3) on Page 12, the acceleration is:

$$\underset{\sim}{a} = \underset{\sim}{a}_{O_2} + \underset{\sim}{\alpha} \times \underset{\sim}{X}_0 + 2\underset{\sim}{\Omega} \times \underset{\sim}{v}_{rot} + \underset{\sim}{\Omega} \times [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0] + \underset{\sim}{a}_{rot}. \quad \textbf{(6)}$$

The term, $2\underset{\sim}{\Omega} \times \underset{\sim}{v}_{rot}$**, is called the Coriolis acceleration.**

In working problems, Equation (2) on Page 11 and Equation (3) on Page 12 are the two most important equations in kinematics, <u>provided that you correctly identify all the terms!</u>

Summary of Kinematic Equations

Position: $\underset{\sim}{r} = R_{0I}\underset{\sim}{E}_I + X_{0i}\underset{\sim}{e}_i$ **(1)**

Velocity: $\underset{\sim}{v} = \underset{\sim}{v}_{O_2} + \underset{\sim}{\Omega} \times \underset{\sim}{X}_0 + \underset{\sim}{v}_{rot}$ **(2)**

$\underset{\sim}{v}_{O_2} = \dfrac{d(R_{0I})}{dt}\underset{\sim}{E}_I$ **is the absolute velocity of the origin of the second coordinate system.** **(3)**

$\underset{\sim}{v}_{rot} = \dfrac{d(X_{0i})}{dt}\underset{\sim}{e}_i$ **is the relative velocity of the point with respect to the origin of the second coordinate system.** **(4)**

$\underset{\sim}{\Omega} = \dfrac{1}{2}e_{ijk}\dfrac{d(a_{jI})}{dt}a_{Ik}^T\underset{\sim}{e}_i$ **is the absolute angular velocity of the second coordinate system.** **(5)**

Acceleration: $\underset{\sim}{a} = \underset{\sim}{a}_{O_2} + \underset{\sim}{\alpha} \times \underset{\sim}{X}_0 + 2\underset{\sim}{\Omega} \times \underset{\sim}{v}_{rot} + \underset{\sim}{\Omega} \times [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0] + \underset{\sim}{a}_{rot}$ **(6)**

$\underset{\sim}{a}_{O_2} = \dfrac{d^2(R_{0I})}{dt^2}\underset{\sim}{E}_I$ **is the absolute acceleration of the origin of the second coordinate system.** **(7)**

$\underset{\sim}{a}_{rot} = \dfrac{d^2(X_{0i})}{dt^2}\underset{\sim}{e}_i$ **is the relative acceleration with respect to the origin of the second coordinate system.** **(8)**

$\underset{\sim}{\alpha} = \dfrac{d(\Omega_i)}{dt}\underset{\sim}{e}_i$ **is the angular acceleration.** **(9)**

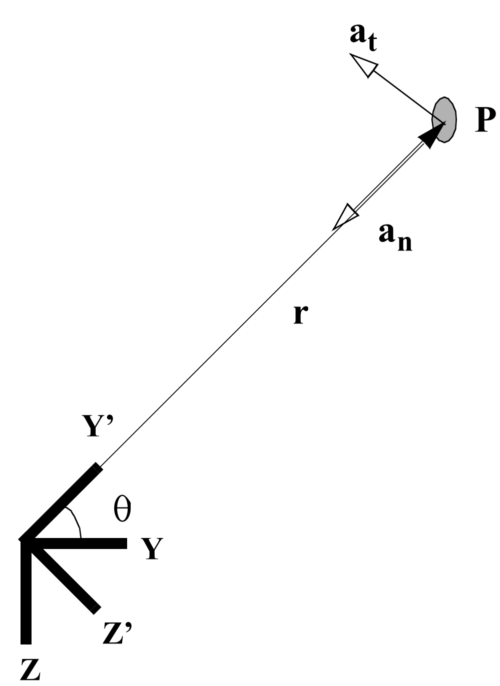
Example: Circular Motion

Recall from earlier that the velocity of an object moving in a circle about a fixed point is: $\underset{\sim}{v}_P = -r\dot{\theta}\underset{\sim}{e}_3'$.

This is a vector, expressed in a rotating frame, which changes in the rotating frame.

The acceleration is the derivative of this vector:

$$\frac{d}{dt}(\underset{\sim}{v}_P) = \frac{d}{dt}(-r\dot{\theta})\underset{\sim}{e}_3' + \underset{\sim}{\Omega} \times [-r\dot{\theta}\underset{\sim}{e}_3'] = [-r\ddot{\theta}\underset{\sim}{e}_3'] - \dot{\theta}\underset{\sim}{e}_1' \times [-r\dot{\theta}\underset{\sim}{e}_3'] = -r\ddot{\theta}\underset{\sim}{e}_3' - r\dot{\theta}^2\underset{\sim}{e}_2'. \qquad (1)$$



By comparison, this could have been computed using equation (6) from page 11 where:

$\underset{\sim}{a}_{O_2} = \underset{\sim}{0}$, $\underset{\sim}{v}_{rot} = \underset{\sim}{0}$, $\underset{\sim}{a}_{rot} = \underset{\sim}{0}$, $\underset{\sim}{\alpha} = -\ddot{\theta}\underset{\sim}{e}_1'$. **This leaves:**

$$\underset{\sim}{a} = \underset{\sim}{\alpha} \times \underset{\sim}{X}_0 + \underset{\sim}{\Omega} \times [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0] = (-\ddot{\theta}\underset{\sim}{e}_1') \times (r\underset{\sim}{e}_2') + (-\dot{\theta}\underset{\sim}{e}_1') \times [(-\dot{\theta}\underset{\sim}{e}_1') \times (r\underset{\sim}{e}_2')] = -r\ddot{\theta}\underset{\sim}{e}_3' - r\dot{\theta}^2\underset{\sim}{e}_2'$$

which agrees with the earlier answer.

Note that the first method requires taking a total time derivative and applying the angular velocity formula. The second method requires the evaluation of two cross products and careful identification of the other terms.

Relative Angular Velocity

In the derivation so far, the rotating reference frame was defined in terms of an absolute reference frame. When the time derivative was taken, the time derivative for the base vectors in the absolute reference frame vanished.

Although this restriction can serve for a large number of problems, it would be convenient to extend the derivation to reference frames which are defined relative non-inertial reference frames.

The derivative of a vector, $X_{0i}\underset{\sim}{e}_i$**, expressed in rotating coordinates was seen to be:**

$$\frac{d(X_{0i}\underset{\sim}{e}_i)}{dt} = \frac{d(X_{0i})}{dt}\underset{\sim}{e}_i + X_{0i}\frac{d(\underset{\sim}{e}_i)}{dt} = [\underset{\sim}{\Omega} \times \underset{\sim}{X}_0] + \frac{d(X_{0i})}{dt}\underset{\sim}{e}_i \qquad \textbf{(1)}$$

where $\underset{\sim}{\Omega} = \Omega_i\underset{\sim}{e}_i = \frac{1}{2}e_{ijk}G_{jk}\underset{\sim}{e}_i$, $G_{ji} = \frac{d(a_{jK})}{dt}a_{Ki}^T = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix}$**, and** $\underset{\sim}{e}_k = a_{kI}\underset{\sim}{E}_I$**.**

What happens when one frame, $\underset{\sim}{e}_{j'}'$**, is rotating with respect to another frame,** $\underset{\sim}{e}_i$**, which is itself rotating with respect to an inertial reference frame?**

The vector can be written in each frame as $X_{0i}\underset{\sim}{e}_i = X_{1j}\underset{\sim}{e}_{j'}'$**.**

Let $\underset{\sim}{e}_{j'}' = b_{j'i}\underset{\sim}{e}_i$ **be the second CTM relation. Since** $\underset{\sim}{e}_k = a_{ki}\underset{\sim}{E}_i$**, it can be seen that:** $\underset{\sim}{e}_{j'}' = b_{j'k}a_{kI}\underset{\sim}{E}_I$**.**

Rotating frame with respect to another rotating frame

The invariant $\underset{\sim}{X} = X_{0i}\underset{\sim}{e}_i = X_{1i'}\underset{\sim}{e}_{i'}$**. The derivative is** $\frac{d\underset{\sim}{X}}{dt} = \frac{d(X_{1i'})}{dt}\underset{\sim}{e}_{i'}' + X_{1i'}\frac{d(\underset{\sim}{e}_{i'}')}{dt}$**.**

The derivative of this unit vector set is:

$$\frac{d(\underset{\sim}{e}_{j'}')}{dt} = \frac{d}{dt}[b_{j'k}a_{kI}\underset{\sim}{E}_I] = \frac{d(b_{j'k})}{dt}a_{kI}\underset{\sim}{E}_I + b_{j'k}\frac{d(a_{kI})}{dt}\underset{\sim}{E}_I = \frac{d(b_{j'k})}{dt}\underset{\sim}{e}_k + b_{j'k}\frac{d(a_{kI})}{dt}\underset{\sim}{E}_I \quad .$$

By definition of the CTMs, $\underset{\sim}{E}_I = a^T_{Im}\underset{\sim}{e}_m$ **and** $\underset{\sim}{e}_k = b^T_{ki'}\underset{\sim}{e}_{i'}'$ **.**

$$\frac{d(\underset{\sim}{e}_{j'}')}{dt} = \frac{d(b_{j'k})}{dt}\underset{\sim}{e}_k + b_{j'k}\frac{d(a_{kI})}{dt}\underset{\sim}{E}_I = \frac{d(b_{j'k})}{dt}b^T_{ki'}\underset{\sim}{e}_{i'}' + b_{j'k}\frac{d(a_{kI})}{dt}a^T_{Im}\underset{\sim}{e}_m \quad .$$

Defining $G_{km} = \frac{d(a_{kI})}{dt}a^T_{Im}$ **and** $H_{j'i'} = \frac{d(b_{j'k})}{dt}b^T_{ki'}$ **,** $\frac{d(\underset{\sim}{e}_{j'}')}{dt} = H_{j'p'}\underset{\sim}{e}_{p'}' + b_{j'k}G_{km}\underset{\sim}{e}_m$ **.**

From earlier work, it can be concluded that: $G_{km} = e_{pkm}\Omega_p$ **and** $H_{j'p'} = e_{k'j'p'}\Phi_{k'}$ **, and therefore that:**

$$\frac{d(\underset{\sim}{e}_{j'}')}{dt} = e_{k'j'p'}\Phi_{k'}\underset{\sim}{e}_{p'}' + e_{pkm}\Omega_p b^T_{kj'}\underset{\sim}{e}_m \quad .$$

(2)

More Angular Velocity

Substituting $\frac{d(\underset{\sim}{e}_{j'}')}{dt}$ **into the derivative of the vector,** $\underset{\sim}{X}$**:**

$$\frac{d\underset{\sim}{X}}{dt} = \frac{d(X_{1i'})}{dt}\underset{\sim}{e}_{i'}' + X_{1j'}[e_{k'j'p'}\Phi_{k'}\underset{\sim}{e}_{p'}' + e_{pkm}\Omega_p b^T_{kj'}\underset{\sim}{e}_m]$$

From earlier, the first term of Equation on Page 18 is $\underset{\sim}{v}_{rot} = \frac{d(X_{1i'})}{dt}\underset{\sim}{e}_{i'}'$**. This is the derivative of the time varying components of the vector, as seen in the second rotating reference frame.**

The second term is $e_{k'j'p'}\Phi_{k'}X_{1j'}\underset{\sim}{e}_{p'}' = \underset{\sim}{\Phi} \times \underset{\sim}{X}$ **.**

The third term is $X_{1j'}e_{pkm}\Omega_p b^T_{kj'}\underset{\sim}{e}_m = e_{pkm}\Omega_p b^T_{kj'}X_{1j'}\underset{\sim}{e}_m = e_{pkm}\Omega_p X_{0k}\underset{\sim}{e}_m = \underset{\sim}{\Omega} \times \underset{\sim}{X}$ **.**

Therefore, $\frac{d\underset{\sim}{X}}{dt} = \underset{\sim}{v}_{rot} + (\underset{\sim}{\Phi} + \underset{\sim}{\Omega}) \times \underset{\sim}{X}$

The total time derivative of a vector expressed in any set of rotating reference frames is the derivative of the components in that reference frame ($\frac{d(X_{1i'})}{dt}\underset{\sim}{e}_{i'}'$**) plus the <u>absolute</u> angular velocity of the reference frame crossed with the rotating vector (**$(\underset{\sim}{\Omega} + \underset{\sim}{\Phi}) \times \underset{\sim}{X}$ **).**

Relative Angular Acceleration

This is the general rule for evaluating a vector expressed in a rotating reference frame.

The absolute angular velocity, denoted $\underset{\sim}{\omega}$, is the sum of the relative angular velocities of all reference frames between the inertial reference frame and the final, rotating reference frame. In the case of two reference frames used in this derivation, $\underset{\sim}{\omega} = \underset{\sim}{\Omega} + \underset{\sim}{\Phi}$.

The angular acceleration is $\underset{\sim}{\alpha} = \frac{d(\underset{\sim}{\omega})}{dt} = \frac{d(\omega_{i'})}{dt}\underset{\sim}{e}'_{i'}$ from Equation (1) on Page 13, where $\underset{\sim}{\Omega}$ has been replaced by $\underset{\sim}{\omega}$ to indicate that this is valid for any number of reference frames tied to an inertial reference frame.

In order to get this expression, the angular velocity was expressed in a single reference frame. Consider if this reference frame, $\underset{\sim}{e}'_{j'}$, were rotating with respect to another rotating frame, $\underset{\sim}{e}_i$, which is referenced to an inertial reference frame, $\underset{\sim}{E}_I$.

In that case, the angular velocity would be $\underset{\sim}{\omega} = \underset{\sim}{\Omega} + \underset{\sim}{\Phi}$ (see page 18).

The CTM relations are $\underset{\sim}{e}_k = a_{kI}\underset{\sim}{E}_I$ and $\underset{\sim}{e}'_{j'} = b_{j'i}\underset{\sim}{e}_i$ and the angular velocities are $\underset{\sim}{\Omega} = \Omega_i\underset{\sim}{e}_i = \frac{1}{2}e_{ijk}G_{jk}\underset{\sim}{e}_i$ and

$$\underset{\sim}{\Phi} = \Phi_{i'}\underset{\sim}{e}'_{i'} = \frac{1}{2}e_{i'j'k'}H_{j'k'}\underset{\sim}{e}'_{i'}$$ **where** $$G_{ji} = \frac{d(a_{jK})}{dt}a^T_{Ki}$$ **and** $$H_{j'i'} = \frac{d(b_{j'k})}{dt}b^T_{ki'}.$$

Recursion in Angular Velocity and Acceleration

The derivative of the angular velocity is

$$\frac{d(\underset{\sim}{\omega})}{dt} = \frac{d(\underset{\sim}{\Omega})}{dt} + \frac{d(\underset{\sim}{\Phi})}{dt} = \frac{d(\Omega_j)}{dt}\underset{\sim}{e}_j + \underset{\sim}{\Omega} \times \underset{\sim}{\Omega} + \frac{d(\Phi_{i'})}{dt}\underset{\sim}{e}'_{i'} + (\underset{\sim}{\Omega} + \underset{\sim}{\Phi}) \times \underset{\sim}{\Phi} = \frac{d(\Omega_j)}{dt}\underset{\sim}{e}_j + \frac{d(\Phi_{i'})}{dt}\underset{\sim}{e}'_{i'} + \underset{\sim}{\Omega} \times \underset{\sim}{\Phi} \quad .$$

When calculating angular velocity for a series of linked coordinate systems, it is useful to calculate the angular velocity and acceleration in terms of the previous system's value.

Let the coordinate systems be denoted by $\underset{\sim}{e}_{(\zeta),i}$, where ζ denotes the number of the coordinate system in the chain. The parentheses around the index indicate that indicial notation rules are not in effect with respect to this index.

For instance, $\underset{\sim}{e}_{(0),i}$ is the first rotating coordinate system relative to the inertial coordinate system, $\underset{\sim}{e}_{(1),i}$ is relative to $\underset{\sim}{e}_{(0),i}$, and so on.

The angular velocity for the ζ^{th} coordinate system relative to the $(\zeta - 1)^{st}$ is denoted by $\underset{\sim}{\Omega}_{(\zeta)}$.

The angular acceleration for the ζ^{th} coordinate system relative to the $(\zeta - 1)^{st}$ is denoted by $\underset{\sim}{\xi}_{(\zeta)} = \frac{d\Omega_{(\zeta),i}}{dt}\underset{\sim}{e}_{(\zeta),i}$.

In this notation, the absolute angular velocity for the ζ^{th} coordinate system is $\underset{\sim}{\omega}_{(\zeta)} = \sum_{k=0}^{\zeta} \underset{\sim}{\Omega}_{(k)}$. This can be written

recursively as $\underset{\sim}{\omega}_{(\zeta+1)} = \underset{\sim}{\Omega}_{(\zeta+1)} + \underset{\sim}{\omega}_{(\zeta)}$ **.**

Since $\underset{\sim}{\alpha}_{(\zeta+1)} = \frac{d\underset{\sim}{\omega}_{(\zeta+1)}}{dt} = \frac{d\underset{\sim}{\Omega}_{(\zeta+1)}}{dt} + \frac{d\underset{\sim}{\omega}_{(\zeta)}}{dt} = \frac{d\underset{\sim}{\Omega}_{(\zeta+1)}}{dt} + \underset{\sim}{\alpha}_{(\zeta)}$ **, it is necessary to determine** $\frac{d\underset{\sim}{\Omega}_{(\zeta+1)}}{dt}$ **.**

Recursion in Angular Acceleration

The relative angular velocity is $\underset{\sim}{\Omega}_{(\zeta)} = \Omega_{(\zeta),i}\underset{\sim}{e}_{(\zeta),i}$ **.**

Hence,

$$\frac{d\underset{\sim}{\Omega}_{(\zeta)}}{dt} = \frac{d\Omega_{(\zeta),i}}{dt}\underset{\sim}{e}_{(\zeta),i} + \Omega_{(\zeta),i}\frac{d(\underset{\sim}{e}_{(\zeta),i})}{dt} = \underset{\sim}{\xi}_{(\zeta)} + \underset{\sim}{\omega}_{(\zeta)} \times \underset{\sim}{\Omega}_{(\zeta)} = \underset{\sim}{\xi}_{(\zeta)} + \left(\sum_{k=0}^{\zeta}\underset{\sim}{\Omega}_{(k)}\right) \times \underset{\sim}{\Omega}_{(\zeta)} = \underset{\sim}{\xi}_{(\zeta)} + \left(\sum_{k=0}^{\zeta-1}\underset{\sim}{\Omega}_{(k)}\right) \times \underset{\sim}{\Omega}_{(\zeta)} .$$

This can be simplified to $\frac{d\underset{\sim}{\Omega}_{(\zeta)}}{dt} = \underset{\sim}{\xi}_{(\zeta)} + \underset{\sim}{\omega}_{(\zeta-1)} \times \underset{\sim}{\Omega}_{(\zeta)}$ **.**

Substituting into the recursion formula:

$$\underset{\sim}{\alpha}_{(\zeta+1)} = \underset{\sim}{\alpha}_{(\zeta)} + \underset{\sim}{\xi}_{(\zeta+1)} + \underset{\sim}{\omega}_{(\zeta)} \times \underset{\sim}{\Omega}_{(\zeta+1)} .$$

To use this formula, both the current value of <u>relative</u> angular velocity, $\underset{\sim}{\Omega}_{(\zeta+1)}$ **, and <u>relative</u> angular acceleration,** $\underset{\sim}{\xi}_{(\zeta+1)} = \frac{d\Omega_{(\zeta+1),i}}{dt}\underset{\sim}{e}_{(\zeta+1),i}$ **, are computed. The previous value of the absolute angular velocity,** $\underset{\sim}{\omega}_{(\zeta)}$**, and the absolute angular acceleration,** $\underset{\sim}{\alpha}_{(\zeta)}$**, are used.**

Degrees of Freedom

When solving a dynamic problem, the concept of degrees of freedom emerges. Both Baruh and Greenwood provide definitions for this concept.

Greenwood (page 229): "The number of degrees of freedom is equal to the number of coordinates which are used to specify the configuration of the system minus the number of independent equations of constraint."

Baruh (page 35): "the minimum number of independent coordinates necessary to describe the configuration of a system. Each constraint applied to a system reduces the number of degrees of freedom by one."

Neither of these definitions provides much physical insight.

Consider a point mass in three dimensional space. Perhaps this is a spherical satellite, nowhere near a gravity well. This body could move arbirarily in three directions, independently. Hence, it has three degrees of freedom. If a student wanted to describe the motion of the body, he would need three separate pieces of information (for instance, x, y, and z coordinates). These three numbers are *generalized coordinates*.

Consider a link which pivots about an axis (such as a pendulum). This link is *constrained* so that it only moves about one axis. It cannot translate in free space. Nor can it rotate about the other two axes. This link would have one Degree of Freedom.

Consider two links (double pendulum). This system has two independent motions which it can undergo (link relative to the ground pivot, second link relative to the first link. This system has two Degrees of Freedom.

The number of DoF is the number of generalized coordinates minus the number of independent constraints.

Constraint Types

Forces and Constraints are closely related.

A Force is actually a theoretical entity. It is the mathematical resultant of real effects, such as surface contact.

Forces can also be "body forces." These forces are resultants from bodies or fields acting upon each other. Examples of these forces are gravity and electromagnetic forces.

Forces can constrain motion, such as a pivot, a slider, or a universal joint. When a force constrains motion, it removes degrees of freedom. The actual value of the force is usually irrelevant, unless you are performing stress analysis to determine whether the constraining body will break.

Constraint forces come in several varieties:

scleronomic or rheonomic

holonomic or non-holonomic

The type of constraint which removes a degree of freedom in the purely kinematic sense is a workless contraint. A workless constraint does not perform any work on the system. This can either occur because the constraint force is zero, the displacement is zero, or the component of the force in the direction of the displacement is zero.

$$dW = \underset{\sim}{F} \bullet d\underset{\sim}{r} = 0$$

Contraint Types

If $q_1, q_2, \ldots, q_n$ **are the** n **generalized coordinates describing a system, then a holonomic contraint has the form:**

$$\phi(q_1, q_2, \ldots, q_n, t) = 0.$$

Generalized coordinates subject to holonomic contraints can usually be reduced to a set of independent generalized coordinates, by using the constraint equations to eliminate variables.

A non-holonomic constraint has the form:

$$\sum_{i=1}^{n} [\alpha_i(q_1, q_2, \ldots, q_n, t)dq_i + \beta_i(q_1, q_2, \ldots, q_n, t)dt] = 0,$$

where α_i **and** β_i **are such that this equation is not integrable.**

These constraints are also rheonomic, because they contain time in the functional expression.

A scleronomic constraint does not contain time. The scleronomic, holonomic expression is:

$$\phi(q_1, q_2, \ldots, q_n) = 0.$$

No Slip Condition

When two surfaces are moving relative to each other, sometimes a "no slip" constraint is imposed. This reduces the number of generalized coordinates by up to three degrees of freedom.

Two points, one on the ground, the other on the body, which are in contact at some point in time, travel the same distance.

This is the same as saying that, instantaneously, the velocity of the point in contact with the ground is the same as the velocity of the ground.

Both velocities must be absolute velocities.

If the velocities are written in the same coordinate system, then:

$$v_{px} = v_{gx} \qquad v_{py} = v_{gy} \qquad v_{pz} = v_{gz} \tag{1}$$

Taking the derivatives of components gives:

$$\dot{v}_{px} = \dot{v}_{gx} \qquad \dot{v}_{py} = \dot{v}_{gy} \qquad \dot{v}_{pz} = \dot{v}_{gz}. \tag{2}$$

Since independent variables usually appear in the second derivative, Equation (2) on Page 25 gives the more useful no slip condition.

Example: Wheel moving in a Straight Line

Consider a wheel of radius, r**, which rolls in a straight line.**

A vector pointing from a fixed reference frame to the center of the wheel is $\underset{\sim}{r}_c = X\underset{\sim}{I}$**.**

A vector pointing from the center to a point on the wheel is $\underset{\sim}{r}_{p/c} = r(\cos\theta\underset{\sim}{I} + \sin\theta\underset{\sim}{J})$**, where** θ **is referenced from the horizontal and increases in the counter-clockwise direction.**

The velocity of a point on the rim of the wheel is $\underset{\sim}{v}_p = \dot{X}\underset{\sim}{I} + r\dot{\theta}((-\sin\theta)\underset{\sim}{I} + \cos\theta\underset{\sim}{J})$**.**

When the point contacts the ground, $\theta = -\frac{\pi}{2}$ **and the absolute velocity of the point, when it contacts the ground, is:**

$$\underset{\sim}{v}_p = \dot{X}\underset{\sim}{I} + r\dot{\theta}\underset{\sim}{I}. \tag{1}$$

At any instant in time, a different point contacts the ground. Each such point has a different reference θ**. The difference between these references is a constant offset,** θ_{offset}**. Because this offset is constant,** $\dot{\theta}$ **is the same for each reference angle and the velocity of the point in contact with the ground is always given by equation (1).**

Since the ground is not moving, $\underset{\sim}{v}_p = (\dot{X} + r\dot{\theta})\underset{\sim}{I} = \underset{\sim}{0}$ **and** $\dot{X} = -r\dot{\theta}$**. Further,** $\ddot{X} = -r\ddot{\theta}$**.**

This is clearly a contraint which relates the two variables.

Wheel Wandering in a Plane

Consider a wheel which can rotate about its point of contact and wander in a two dimensional plane. The position of the center of the wheel is now $\underset{\sim}{r}_c = X\underset{\sim}{I} + Z\underset{\sim}{K}$**. Define the** $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ **coordinate system which initially coincides with the inertial coordinate system for the previous example, but moves with the center of the wheel. This system has angular velocity,** $\underset{\sim}{\omega} = \dot{\gamma}\underset{\sim}{j}$ **.**

A point on the rim of the wheel must be expressed in the rotating coordinate system: $\underset{\sim}{r}_{p/c} = r(\cos\theta\underset{\sim}{i} + \sin\theta\underset{\sim}{j})$**.**

The absolute velocity of the point is: $\underset{\sim}{v}_p = \dot{X}\underset{\sim}{I} + \dot{Z}\underset{\sim}{K} + r\dot{\theta}(-\sin\theta\underset{\sim}{i} + \cos\theta\underset{\sim}{j}) - r\dot{\gamma}\cos\theta\underset{\sim}{k}$**.**

The same argument about θ_{offset} **holds and the velocity of the point in contact with the ground occurs when** $\theta = -\frac{\pi}{2}$ **(because of the initial reference set up for this particular point):**

$$\underset{\sim}{v}_p = \dot{X}\underset{\sim}{I} + \dot{Z}\underset{\sim}{K} + r\dot{\theta}\underset{\sim}{i}. \quad \textbf{(1)}$$

Since $\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix}$**,** $\underset{\sim}{v}_p = (\dot{X}\cos\gamma - (\dot{Z}\sin\gamma) + r\dot{\theta})\underset{\sim}{i} + (\dot{Z}\cos\gamma + \dot{X}\sin\gamma)\underset{\sim}{k}$**.**

The condition to prevent rolling slip is $\dot{X}\cos\gamma - \dot{Z}\sin\gamma = -r\dot{\theta}$**.**

The condition to prevent side slip is $\dot{Z}\cos\gamma + \dot{X}\sin\gamma = 0$**.**

These two conditions can be solved to give two variables in terms of the third.

Dynamics of a Particle

Now that we can describe how vectors move, let's do something.

Consider a point particle with mass, m. This particle's motion is described by its position vector, $\underset{\sim}{r}$**.**

Position requires three generalized coordinates to represent it uniquely. Rotation of the particle does not mean anything since the particle is a point-mass.

Since this particle has mass, it has momentum: $\underset{\sim}{p} = m\underset{\sim}{v} = m\frac{d(\underset{\sim}{r})}{dt}$**.**

Assume that several forces are acting on this particle. We can apply Newton's Second Law: "The rate of change of momentum is equal to the sum of the applied forces."

$$\sum \underset{\sim}{F} = \frac{d(\underset{\sim}{p})}{dt} \qquad \textbf{(1)}$$

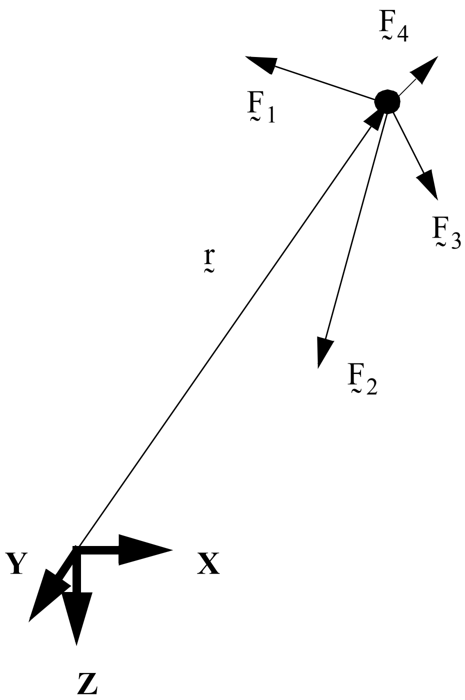
This is sometimes called the conservation of linear momentum.

Inserting momentum and evaluating the derivatives, assuming that mass is constant, yields:

$$\sum \underset{\sim}{F} = m\frac{d^2(\underset{\sim}{r})}{dt^2} = m\underset{\sim}{a}. \qquad \textbf{(2)}$$

Angular Momentum about a Fixed Point

Consider the first moment of momentum: $\underset{\sim}{H}_o = \underset{\sim}{r} \times \underset{\sim}{p}$. This vector is called the angular momentum, and it is referenced to the point of origin of the position vector, $\underset{\sim}{r}$. The origin, O, must be fixed! It is not sufficient for that point to be inertial. This point is not necessarily the origin of the reference frame where $\underset{\sim}{p}$ was defined, but the same origin will be used for both systems to avoid confusion.



The time derivative of the angular momentum is:

$$\frac{d\underset{\sim}{H}_o}{dt} = \frac{d(\underset{\sim}{r} \times \underset{\sim}{p})}{dt} = \left(\frac{d\underset{\sim}{r}}{dt}\right) \times \underset{\sim}{p} + \underset{\sim}{r} \times \left(\frac{d\underset{\sim}{p}}{dt}\right). \tag{1}$$

Noting that $\sum \underset{\sim}{F} = \left(\frac{d\underset{\sim}{p}}{dt}\right)$, $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt}$, and $\underset{\sim}{p} = m\underset{\sim}{v}$, this reduces to:

$$\frac{d\underset{\sim}{H}_o}{dt} = \underset{\sim}{v} \times (m\underset{\sim}{v}) + \underset{\sim}{r} \times (\sum \underset{\sim}{F}). \tag{2}$$

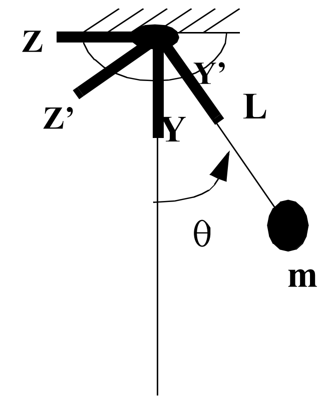
The first term on the right hand side is a vector crossed with itself, and therefore the cross product is zero. The second term is the first moment, $\underset{\sim}{M}_o$, of the forces acting on the particle.

The Conservation of Angular Momentum is: $\frac{d}{dt}(\underset{\sim}{H}_o) = \sum \underset{\sim}{M}_o$.

Example: Simple Pendulum

Consider a pendulum anchored at one end and with a mass, m, connected to the anchor via a massless string of length, L.

From the circular motion example, we know the acceleration of the particle is:

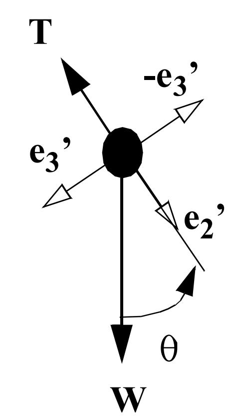


$$\underset{\sim}{a}_P = -L\ddot{\theta}\underset{\sim}{e}_3' - L\dot{\theta}^2\underset{\sim}{e}_2' \qquad (1)$$

Consider the mass alone. Cut away the string and replace its effects with the vector force, $\underset{\sim}{T}$.

Since the string is massless, the tension must act along the line of the string. Hence, only its magnitude is unknown, and: $\underset{\sim}{T} = -T\underset{\sim}{e}_2'$.

The only other force acting on the mass is due to gravity, which acts in the $\underset{\sim}{E}_3$-direction,



$$\underset{\sim}{W} = mg\underset{\sim}{E}_2 = mg(\cos\theta\underset{\sim}{e}_2' + \sin\theta\underset{\sim}{e}_3') \quad . \qquad (2)$$

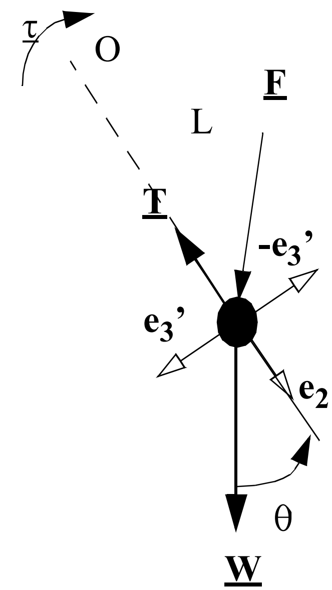
The sum of forces acting on this particle is: $\sum\underset{\sim}{F} = \underset{\sim}{T} + \underset{\sim}{W} = (mg\cos\theta - T)\underset{\sim}{e}_2' + mg\sin\theta\underset{\sim}{e}_3'$ **.**

By Newton's Second Law: $(mg\cos\theta - T)\underset{\sim}{e}_2' + mg\sin\theta\underset{\sim}{e}_3' = m(-L\ddot{\theta}\underset{\sim}{e}_3' - L\dot{\theta}^2\underset{\sim}{e}_2')$ **.**

Since all we want to know is the equation of motion, we don't care about the tension.

Equating the $\underset{\sim}{e}_3'$-component yields: $mg\sin\theta = -mL\ddot{\theta}$ **. The Equation of Motion is:** $m(L\ddot{\theta} + g\sin\theta) = 0$ **Example: Simple Pendulum via Angular Momentum**

Consider the simple pendulum of the previous example. Consider a couple of new forces: a force on the mass $F = F_2\underset{\sim}{e}'_2 + F_3\underset{\sim}{e}'_3$**, and a motor torque at the pivot,** $\underset{\sim}{\tau} = \tau\underset{\sim}{e}'_1$**.**



The momentum of the mass is $\underset{\sim}{p} = m({}^P\underset{\sim}{v}) = -mL\dot{\theta}\underset{\sim}{e}_3'$**. The angular momentum of the particle about O is:** $\underset{\sim}{H}_o = \underset{\sim}{r}\times\underset{\sim}{p} = (L\underset{\sim}{e}'_2)\times(-mL\dot{\theta}\underset{\sim}{e}'_3) = -mL^2\dot{\theta}\underset{\sim}{e}'_1$ **.**

The time derivative of the angular momentum is:

$$\frac{d}{dt}(\underset{\sim}{H}_o) = \frac{d}{dt}(-mL^2\dot{\theta})\underset{\sim}{e}'_1 + (-mL^2\dot{\theta})\frac{d}{dt}(\underset{\sim}{e}'_1) = -mL^2\ddot{\theta}\underset{\sim}{e}'_1 + (-\dot{\theta}\underset{\sim}{e}_1')\times(-mL^2\dot{\theta})\underset{\sim}{e}'_1 = -mL^2\ddot{\theta}\underset{\sim}{e}'_1 \qquad \textbf{(3)}$$

The sum of the moments due to external forces is:

$$\sum\underset{\sim}{M}_o = (L\underset{\sim}{e}'_2)\times((mg\cos\theta - T)\underset{\sim}{e}_2' + mg\sin\theta\underset{\sim}{e}_3' + F_2\underset{\sim}{e}'_2 + F_3\underset{\sim}{e}'_3) + \tau\underset{\sim}{e}'_1 = (mgL\sin\theta + F_3L + \tau)\underset{\sim}{e}'_1 \qquad \textbf{(4)}$$

The equations of motion are: $(mgL\sin\theta + F_3L + \tau)\underset{\sim}{e}'_1 = -mL^2\ddot{\theta}\underset{\sim}{e}'_1$**.**

One component of this vector equation is non-zero. The equation of motion for the independent variable, θ**, is:**

$$(F_3L + \tau) = -mL^2\ddot{\theta} - mgL\sin\theta. \qquad \textbf{(5)}$$

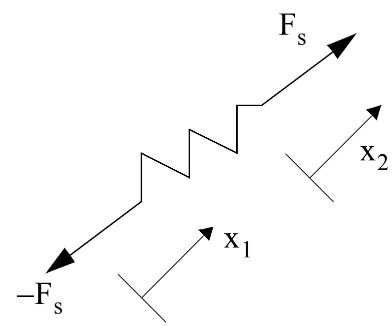
Since θ **is defined as positive clockwise, the right hand side of this equation is the negative of the traditional result.**

Types of Forces - Springs and Dashpots

Two special elements which arise in dynamics are the spring and the dashpot. These are linear elements.

A spring generates a force at each end which is proportional to its compression.

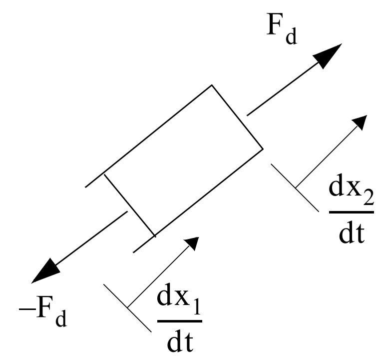
$$F_s = -k(x_2 - x_1).$$



The direction is determined by the direction in which the spring is pointed and must be defined when the problem is set up. The k is the spring constant (in N/m). The negative sign indicates that, for a compression ($x_2 < x_1$), the spring exerts a positive force (i.e. to overcome the compression).

A dashpot (or shock absorber) is an element that exerts a force which is proportional to the relative velocity of the two ends of the element.

$$F_d = -c\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$



Types of Forces - Friction

Dry Friction or Coulomb Friction is a non-linear force.

When two surfaces are not moving relative to each other, they adhere to the no-slip condition. In this case, the contact friction is linear and is a workless constraint.

However, if the two surfaces start to slip, then they are no longer constrained and the friction starts to do work. It them becomes an external force.

When it becomes an external force, it functions in a non-linear manner.

When friction is holding surfaces in no slip, its value is determined by calculating the constraint force.

When that value exceeds a maximum, as determined by the coefficient of static friction (μ_s) and the normal force between the two surfaces (N), then slip occurs and the friction value maintains the value determined by the coefficient of kinetic friction (μ_k) and the normal force.

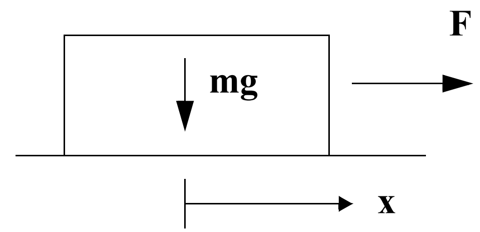
$$\begin{array}{ll} (F_f)_{required} \leq \mu_s N \Rightarrow F_f = (F_f)_{required} & (\text{no slip}) \\ (F_f)_{required} > \mu_s N \Rightarrow F_f = \mu_k N & (\text{ slip}) \end{array} . \qquad \textbf{(1)}$$

The constraint condition relates to the magnitude of the force. The direction is determined by the direction of motion of each surface, relative to the other.

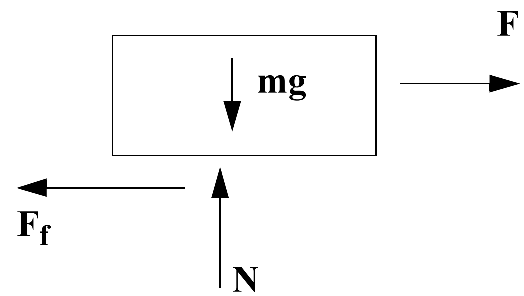
In order for slip to occur, the surfaces must be moving (relatively) in opposite directions, and the friction force maintains Newton's third law. The direction of the friction force opposes the motion.

Types of Forces - Friction (One Degree of Freedom)

The simplest example of sliding friction can be seen from the block (point mass) being pulled by a horizontal force.



Free Body Diagram



The sum of the forces in the y direction yields $N = mg$.

The sum of the forces in the x direction yields $F - F_f = m\ddot{x}$.

If the block is not moving, then $\ddot{x} = 0$ and $F_f = F$.

When $F > \mu_s mg$, the static friction has been broken and the block slides.

The EOM becomes $\ddot{x} = \frac{F}{m} - \mu_k g$.

Types of Forces - Friction (General)

In the general case, both surfaces will be moving and the normal force will vary with both time and with the coordinates of the system.

The magnitude of the force will remain as on Equation (1) on Page 33. However, the direction will depend on the relative velocity between the surfaces.

Define the absolute velocity of one surface as $\underset{\sim}{v}_A$ and for the other surface, $\underset{\sim}{v}_B$.

The relative velocity of the surfaces is $\underset{\sim}{v}_r = \underset{\sim}{v}_B - \underset{\sim}{v}_A$.

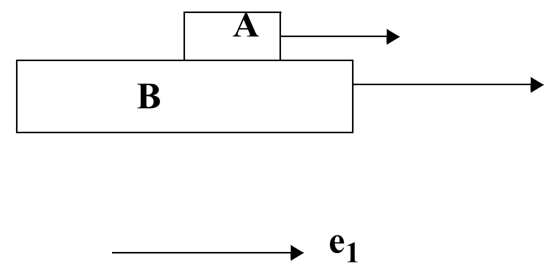
The direction of the friction force exerted by surface A on surface B is $\underset{\sim}{e}_{f,\ A\ on\ B} = -\frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|}$.

Consequently, the direction of the friction force exerted by surface B on surface A is $\underset{\sim}{e}_{f,\ B\ on\ A} = \frac{\underset{\sim}{v}_r}{\|\underset{\sim}{v}_r\|}$.

In reality, the direction of the friction force can be determined from the no-slip friction force. Once sliding has occured, the no-slip force has been exceeded. The direction of the kinetic friction force will be the same as the direction of the no slip constraint force.

Because the no slip force opposes the motion, and this is strictly encapsulated in the mathematics of no slip, the direction of the actual friction force will be the same.

Two Sliding Blocks



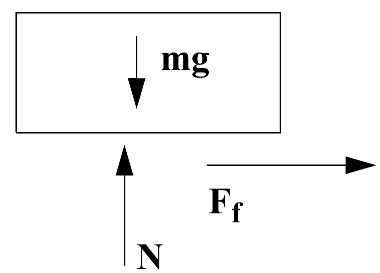
Consider two blocks, one sitting on top of the second. The first block is moving with constant velocity, $\underset{\sim}{v}_B = v\underset{\sim}{e}_1$**.**

The second block is initially moving at $\underset{\sim}{v}_A = a\underset{\sim}{e}_1$**.**

Both blocks have mass, m.

What is the equation of motion describing the second block?

Free Body Diagram



The direction of the friction force acting on block A is $\underset{\sim}{e}_{f,\ B\ on\ A} = \frac{v - \dot{x}}{|v - \dot{x}|}\underset{\sim}{e}_1$**.**

Since there is sliding, the equation of motion, initially, is: $\ddot{x} = \mu_k g\left(\frac{v - \dot{x}}{|v - \dot{x}|}\right)$**. This will continue until block A's velocity reaches the same value as block B's velocity.**

It can be seen that the EOM is a non-linear function of the state.

Note also that $x, \dot{x}, \ddot{x}$ **are referenced to the starting position and are not referenced to the bottom block.**

State Space

One of the things which is done with dynamical equations is to simulate them. In this class, Matlab and the ode45 function are used.

Alas, ode45 only simulates first order differential equations! And all dynamical equations are second order!

What to do? Fortunately, ode45 will simulate vector equations.

There is a technique to convert ordinary differential equations of higher than first order into a vector set of first order ordinary differential equations. This representation is called the state space representation.

Consider the second order ODE: $\ddot{\theta} = g(\dot{\theta}, \theta)$.

Define two new variables: $\begin{matrix} x_1 = \theta \\ x_2 = \dot{\theta} \end{matrix}$.

Taking the derivative of these variables yields:

$$\begin{matrix} \dot{x}_1 = \dot{\theta} = x_2 \\ \dot{x}_2 = \ddot{\theta} = g(\dot{\theta}, \theta) = g(x_2, x_1) \end{matrix}.$$

This new set of equations is a system of first order equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g(x_2, x_1) \end{bmatrix}$$

State Space

If you have a vector of second order equations, state space also works.

Consider: $\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} g(\dot{\theta}, \theta, \dot{x}, x) \\ h(\dot{\theta}, \theta, \dot{x}, x) \end{bmatrix}$**. Define:** $\begin{array}{l} x_1 = \theta \\ x_2 = x \\ x_3 = \dot{\theta} \\ x_4 = \dot{x} \end{array}$**. Taking derivatives yields:**

$$\begin{aligned} \dot{x}_1 &= \dot{\theta} = x_3 \\ \dot{x}_2 &= \dot{x} = x_4 \\ \dot{x}_3 &= \ddot{\theta} = g(\dot{\theta}, \theta, \dot{x}, x) = g(x_1, x_2, x_3, x_4) \\ \dot{x}_4 &= \ddot{x} = h(\dot{\theta}, \theta, \dot{x}, x) = h(x_1, x_2, x_3, x_4) \end{aligned}.$$

Collecting this in matrix form yields:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(x_1, x_2, x_3, x_4) \\ h(x_1, x_2, x_3, x_4) \end{bmatrix}.$$

State Space

A pattern begins to emerge.

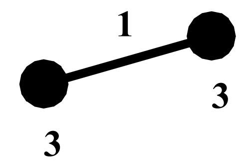
If $[y]$ **is the vector of independent variables in the dynamic system, and** $[x] = \begin{bmatrix} [y] \\ [\dot{y}] \end{bmatrix}$ **is the state space vector such that** $[\dot{x}] = \begin{bmatrix} [\dot{y}] \\ [\ddot{y}] \end{bmatrix}$**, and if the dynamics equation is** $[M][\ddot{y}] = [g([\dot{y}],[y])]$**, then the state space equation is:**

$$\begin{bmatrix} [I] & [0] \\ [0] & [A] \end{bmatrix}[\dot{x}] = \begin{bmatrix} [0] & [I] \\ [0] & [0] \end{bmatrix}[x] + \begin{bmatrix} [0] \\ [g([\dot{y}],[y])] \end{bmatrix}$$

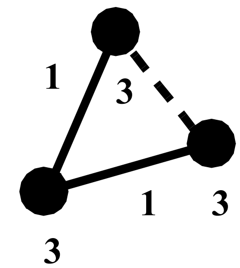
Rigid Body

A rigid body is an infinite system of particles, each with infinitessimal mass, dm, which is constrained so that each mass does not move relative to the other particles. Hence, it cannot deform under load or vibrate.

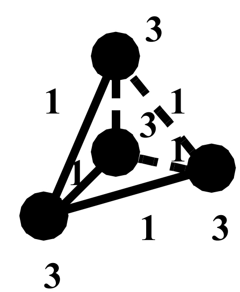
A system of n particles has 3n degrees of freedom. The rigid body, however, imposes constraints on the degrees of freedom.



Consider a rigid body composed of two particles (a linear rigid body). Each particle has three degrees of freedom. However, there is one constraint on those degrees of freedom. The actual degrees of freedom of this system is 6-1=5.



Consider a rigid body composed of three points (a planar rigid body). This system has 9 DOF. However, there are three independent constraints (the lengths between each point) for this system. The actual DOF is 9-3=6.



Add a fourth, out of plane, point to this system. Again, there are 3 additional DOF for the point and only three additional independent constraints. The system DOF is 12 - 6 = 6.

Any further addition of points after the initial three does not yield any more or fewer system DOF. Each new point adds three DOF and three independent constraints.

Hence, the infinite collection of particles has 6 DOF.

Euler Angles

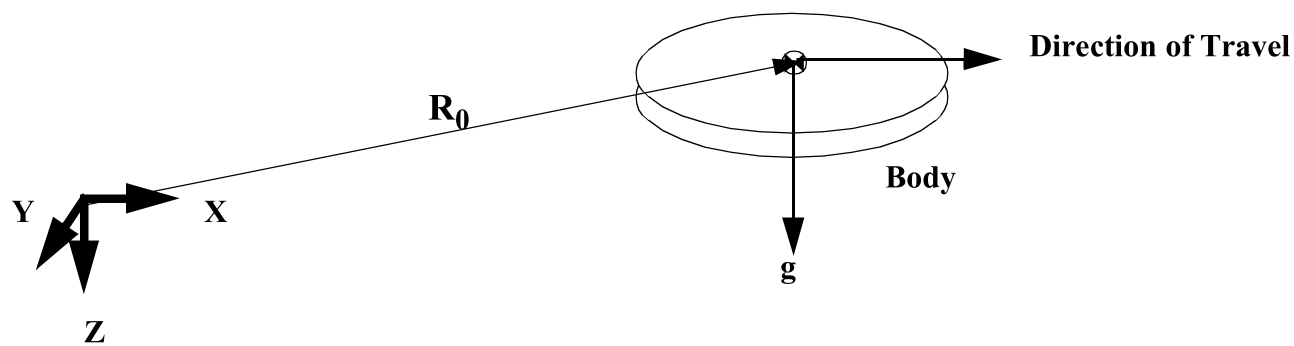
To describe the position of an arbitray rigid body, six independent coordinates need to be specified: 3 translational and 3 rotational.

The three translational degrees of freedom can be specified by any convenient set of coordinates, such as cartesian, cylindrical, or spherical.

There are many different Euler angle formulations (see Greenwood pp. 333-335 or Baruh pp. 368-373, 419). Only one will be described in this course, which comes from aircraft applications.

Consider a set of axes embedded in the body of the walking robot and their unit vectors ($\underset{\sim}{E}_I$) as shown in the figure. The $\underset{\sim}{E}_1$ vector points in the direction of travel. The $\underset{\sim}{E}_3$ vector points in the direction of gravity. The $\underset{\sim}{E}_2$ vector is perpendicular to $\underset{\sim}{E}_1$ and $\underset{\sim}{E}_3$, such that $\underset{\sim}{E}_I$ is a right-handed coordinate system.

These vectors are initially aligned in these orientations. As the body translates and rotates, these vectors remain fixed. However, three other sets of coordinates will be defined to account for the rotation.



Euler Angles

The first rotation is made about the Z axis, and results in the lower case system ($\underset{\sim}{e}_{(1)i}$). The angle of rotation is ψ. This rotation is called the <u>heading</u>.

The CTM for this rotation is: $\underset{\sim}{e}_{(1)i} \bullet \underset{\sim}{E}_J = \Psi_{iJ} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The second rotation takes the lower case system to the primed system ($\underset{\sim}{e}_{(2)k}$). An attitude rotation, θ, is made about the y axis.

The CTM for this rotation is: $\underset{\sim}{e}_{(2)k} \bullet \underset{\sim}{e}_{(1)i} = \Theta_{ki} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$.

Finally, bank rotation, ϕ, is made about the x' axis. This generates the $\underset{\sim}{e}_{(2)m}$ system.

The CTM for this rotation is: $\underset{\sim}{e}_{(2)m} \bullet \underset{\sim}{e}_{(2)k} = \Phi_{mk} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$.

Euler Angle Angular Velocity

Derivatives of vectors expressed in Euler Angle coordinates must be evaluated. The absolute angular velocity of the total system must be known.

The system angular velocity is the sum of the relative angular velocities determined by each frame. Hence,

$$\underset{\sim}{\omega}_{\text{Euler}} = \underset{\sim}{\Omega}_{\Psi} + \underset{\sim}{\Omega}_{\Theta} + \underset{\sim}{\Omega}_{\Phi}.$$

The relative angular velocities are determined from each of the CTMs.

$$G_{\Psi;ik} = \frac{d}{dt}(\Psi_{iJ})\Psi^{T}_{Jk} = \dot{\psi}\begin{bmatrix} -\sin\psi & \cos\psi & 0 \\ -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \dot{\psi}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\underset{\sim}{\Omega}_{\Psi} = \underset{\sim}{\Omega}_{(1)} = \dot{\psi}\underset{\sim}{e}_{(1)3}$.

$$G_{\Theta;ki} = \frac{d}{dt}(\Theta_{kj})\Theta^{T}_{ji} = \dot{\theta}\begin{bmatrix} -\sin\theta & 0 & -\cos\theta \\ 0 & 0 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \dot{\theta}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Hence, $\underset{\sim}{\Omega}_{\Theta} = \underset{\sim}{\Omega}_{(2)} = \dot{\theta}\underset{\sim}{e}_{(2)2}$.

$$G_{\Phi;mk} = \frac{d}{dt}(\Phi_{mn})\Phi^{T}_{nk} = \dot{\phi}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & \cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \dot{\phi}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Hence, $\underset{\sim}{\Omega}_{\Phi} = \underset{\sim}{\Omega}_{(3)} = \dot{\phi}\underset{\sim}{e}_{(3)1}$.

Angular Acceleration for Euler Angles

The relative angular accelerations are:

$$\underset{\sim}{\xi}_{\Psi} = \underset{\sim}{\xi}_{(1)} = \ddot{\psi}\underset{\sim}{e}_{(1)3} \ .$$

$$\underset{\sim}{\xi}_{\Theta} = \underset{\sim}{\xi}_{(2)} = \ddot{\theta}\underset{\sim}{e}_{(2)2}$$

$$\underset{\sim}{\xi}_{\Phi} = \underset{\sim}{\xi}_{(3)} = \ddot{\phi}\underset{\sim}{e}_{(3)1} \ .$$

The absolute angular accelerations are:

$$\underset{\sim}{\alpha}_{\Psi} = \underset{\sim}{\alpha}_{(1)} = \underset{\sim}{\xi}_{(1)} = \ddot{\psi}\underset{\sim}{e}_{(1)3}$$

$$\underset{\sim}{\alpha}_{\Theta} = \underset{\sim}{\alpha}_{(2)} = \underset{\sim}{\alpha}_{(1)} + \underset{\sim}{\xi}_{(2)} + \underset{\sim}{\omega}_{(1)} \times \underset{\sim}{\Omega}_{(2)} = \ddot{\psi}\underset{\sim}{e}_{(1)3} + \ddot{\theta}\underset{\sim}{e}_{(2)2} + \dot{\psi}\underset{\sim}{e}_{(1)3} \times \dot{\theta}\underset{\sim}{e}_{(2)2}$$

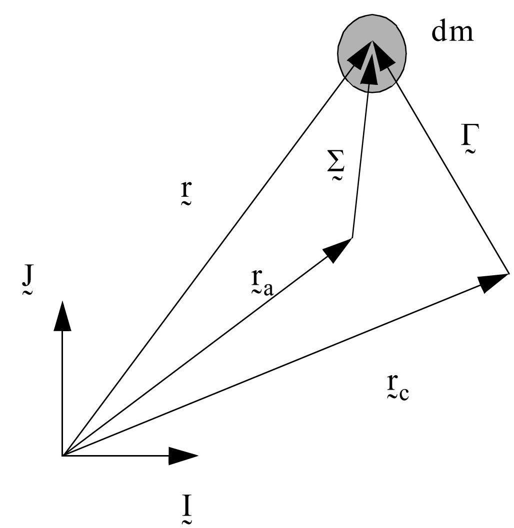
$$\underset{\sim}{\alpha}_{\Phi} = \underset{\sim}{\alpha}_{(3)} = \underset{\sim}{\alpha}_{(2)} + \underset{\sim}{\xi}_{(3)} + \underset{\sim}{\omega}_{(2)} \times \underset{\sim}{\Omega}_{(3)} = \ddot{\psi}\underset{\sim}{e}_{(1)3} + \ddot{\theta}\underset{\sim}{e}_{(2)2} + \dot{\psi}\underset{\sim}{e}_{(1)3} \times \dot{\theta}\underset{\sim}{e}_{(2)2} + \ddot{\phi}\underset{\sim}{e}_{(3)1} + (\dot{\psi}\underset{\sim}{e}_{(1)3} + \dot{\theta}\underset{\sim}{e}_{(2)2}) \times \dot{\phi}\underset{\sim}{e}_{(3)1}$$

Dynamics of a Rigid Body

Describe each point in the rigid body by the vector, $\underset{\sim}{r} = X_i \underset{\sim}{E}_i$. This vector is referenced to an inertial point. For the angular momentum argument which will come later, this point must be fixed.

Consider a reference frame, $\underset{\sim}{e}_i$, moving with a point, $\underset{\sim}{r}_a$, in the rigid body and rotating with the body so that each point in the body appears to be fixed with respect to that point. Each point in the rigid body, $\underset{\sim}{\Sigma} = y_i \underset{\sim}{e}_i$, will appear to be motionless when seen from that point.

Consider that each differential chunk of mass has <u>internal</u> forces exerted by the other bits of mass acting on it. These are the forces which constrain the masses to maintain their positions with respect to each other. Denote each force by $\underset{\sim}{f}_j(y_i)$, which is the force exerted by the jth particle acting on the ith particle.



Coordinate system for Rigid Body

The forces are acting such that $\underset{\sim}{f}_j(y_i) = -\underset{\sim}{f}_i(y_j)$. As a consequence of this assumption, $\underset{\sim}{f}_i(y_i) = 0$ (i.e., the particle cannot exert a force upon itself).

Equation of Motion for a System of Particles

Each particle has an underline external force, $\underset{\sim}{F}(y_i)$, acting on it. This force can be equivalent to the sum of all unbalanced external forces.

The equation of motion for the i^{th} particle is: $\underset{\sim}{F}(y_i) + \sum_{j=1}^{n} \underset{\sim}{f}_j(y_i) = (dm)\frac{d^2(\underset{\sim}{r})}{dt^2}$ **.**

Summing these equations for the set of particles:

$$\sum_{i=1}^{n}\left(\underset{\sim}{F}(y_i) + \sum_{j=1}^{n}\underset{\sim}{f}_j(y_i)\right) = \sum_{i=1}^{n}\left((dm)\frac{d^2(\underset{\sim}{r})}{dt^2}\right) = \frac{d^2}{dt^2}\left(\sum_{i=1}^{n}\underset{\sim}{r}(dm)\right) \quad \textbf{(1)}$$

The first term on the left can be simplified by noting that: $\sum_{i=1}^{n}\sum_{j=1}^{n}\underset{\sim}{f}_j(y_i) = \begin{matrix} 0 & \underset{\sim}{f}_1(y_{2i}) & \dots & \underset{\sim}{f}_1(y_{ni}) \\ \underset{\sim}{f}_2(y_{1i}) & 0 & \dots & \underset{\sim}{f}_2(y_{ni}) \\ \dots & \dots & \dots & \dots \\ \underset{\sim}{f}_n(y_{1i}) & \underset{\sim}{f}_n(y_{2i}) & \dots & 0 \end{matrix} = 0$ **.**

The sum of applied, external forces, $\underset{\sim}{F} = \sum_{i=1}^{n}\underset{\sim}{F}_i$ **.**

Center of Mass

Usually, most of these forces will be zero and a finite number of external forces is applied. However, for some circumstances (usually body forces such as gravity), there may be an infinite number of infinitessimal forces which must be resolved into a finite resultant.

$$\underset{\sim}{F} = \frac{d^2}{dt^2}\left(\sum_{i=1}^{n} \underset{\sim}{r}(dm)\right) \quad \mathbf{(1)}$$

The mass can be represented by $dm = \rho(y_i)dV$ **.**

The sum becomes $\sum_{i=1}^{n} \underset{\sim}{r}(dm) = \sum_{i=1}^{n} \underset{\sim}{r}(y_i)\rho(y_i)dV \rightarrow \int_V \underset{\sim}{r}(y_i)\rho(y_i)dV$ **.**

The total mass of all the particles is $m = \int_V \rho(y_i)dV$ **.**

The center of mass is defined as $\underset{\sim}{r}_c = \frac{1}{m}\int_V \underset{\sim}{r}(y_i)\rho(y_i)dV$ **.**

Then, the result for Newton's Laws for a rigid body become:

$$\underset{\sim}{F} = m\frac{d^2(\underset{\sim}{r}_c)}{dt^2} \text{ .}$$

Angular Momentum of a System of Particles

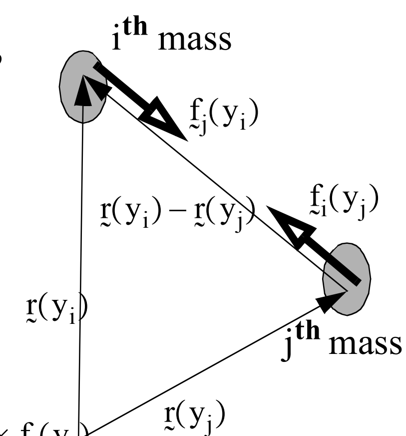
Let the origin of the coordinate system be fixed in space. It is not sufficient for this point to be inertial. Let the fixed point in the body be the center of mass. In that case, $\underset{\sim}{r} = \underset{\sim}{r}_c + y_i \underset{\sim}{e}_i$ **.**

The angular momentum of each particle with respect to this point is defined as: $\underset{\sim}{H}(y_i) = \underset{\sim}{r} \times (dm)\frac{d\underset{\sim}{r}}{dt}$ **.**

The derivative of the angular momentum is:

$$\frac{d\underset{\sim}{H}(y_i)}{dt} = \frac{d\underset{\sim}{r}}{dt} \times (dm)\frac{d\underset{\sim}{r}}{dt} + \underset{\sim}{r} \times (dm)\frac{d^2\underset{\sim}{r}}{dt^2} = \underset{\sim}{r} \times (dm)\frac{d^2\underset{\sim}{r}}{dt^2} .$$

Using the EOM for the mass, $\underset{\sim}{F}(y_i) + \sum_{j=1}^{n} \underset{\sim}{f}_j(y_i) = (dm)\frac{d^2(\underset{\sim}{r})}{dt^2}$,



$$\frac{d\underset{\sim}{H}(y_i)}{dt} = \underset{\sim}{r} \times \left[\underset{\sim}{F}(y_i) + \sum_{j=1}^{n} \underset{\sim}{f}_j(y_i)\right] .$$

Summing up the angular momenta yields the angular momentum for the entire body,

$$\frac{d(\underset{\sim}{H}_o)}{dt} = \sum_{i=1}^{n} \underset{\sim}{r} \times \left[\underset{\sim}{F}(y_i) + \sum_{j=1}^{n} \underset{\sim}{f}_j(y_i)\right] = \sum_{i=1}^{n} \underset{\sim}{r} \times \underset{\sim}{F}(y_i) + \sum_{i=1}^{n} \underset{\sim}{r}(y_i) \times \left[\sum_{j=1}^{n} \underset{\sim}{f}_j(y_i)\right] = \underset{\sim}{M}_o + \sum_{i=1}^{n}\sum_{j=1}^{n} \underset{\sim}{r}(y_i) \times \underset{\sim}{f}_j(y_i)$$

For each pair of forces,

$$\underset{\sim}{r}(y_i) \times \underset{\sim}{f}_j(y_i) + \underset{\sim}{r}(y_j) \times \underset{\sim}{f}_i(y_j) = \underset{\sim}{r}(y_i) \times \underset{\sim}{f}_j(y_i) - \underset{\sim}{r}(y_j) \times \underset{\sim}{f}_j(y_i) = [\underset{\sim}{r}(y_i) - \underset{\sim}{r}(y_j)] \times \underset{\sim}{f}_j(y_i) .$$

From the figure, it can be seen that $\underset{\sim}{r}(y_i) - \underset{\sim}{r}(y_j)$ **is in the same direction as** $\underset{\sim}{f}_j(y_i)$**.**

Angular Momentum of a System of Particles

Whenever two vectors have the same direction and are crossed, the result is zero.

Thus, $\sum_{i=1}^{n}\sum_{j=1}^{n}\underset{\sim}{r}(y_i) \times \underset{\sim}{f}_j(y_i) = 0$ **.**

The equations relating derivative of the angular momentum and moments of forces is:

$$\frac{d(\underset{\sim}{H}_o)}{dt} = M_o$$

where $M_o = \sum_{i=1}^{n}\underset{\sim}{r}(y_i) \times \underset{\sim}{F}(y_i)$ **.**

<u>**The time rate of change of angular momentum with respect to a fixed point in space is equal to the sum of the moments of the external forces about that point.**</u>

Angular Momentum about Center of Mass

Returning to $\frac{d\underset{\sim}{H}(y_i)}{dt} = \underset{\sim}{r} \times (dm)\frac{d^2\underset{\sim}{r}}{dt^2}$ **and the time derivative of angular momentum for the system**

$$\frac{d(\underset{\sim}{H}_o)}{dt} = \sum_{i=1}^{n} \frac{d\underset{\sim}{H}(y_i)}{dt} = \sum_{i=1}^{n} \underset{\sim}{r} \times (dm)\frac{d^2\underset{\sim}{r}}{dt^2} \rightarrow \int_V \underset{\sim}{r} \times (\rho(y_i))\frac{d^2\underset{\sim}{r}}{dt^2}dV$$

The position of each particle can be related to the center of mass by: $\underset{\sim}{r} = \underset{\sim}{r}_c + \underset{\sim}{\Gamma}$ **where** $\underset{\sim}{\Gamma} = y_i\underset{\sim}{e}_i$ **,**

$$\frac{d(\underset{\sim}{H}_o)}{dt} = \int_V (\underset{\sim}{r}_c + \underset{\sim}{\Gamma}) \times (\rho(y_i))\frac{d^2(\underset{\sim}{r}_c + \underset{\sim}{\Gamma})}{dt^2}dV \quad .$$

The first term is $\int_V \underset{\sim}{r}_c \times \rho(y_i)\frac{d^2\underset{\sim}{r}_c}{dt^2}dV = m\underset{\sim}{r}_c \times \frac{d^2\underset{\sim}{r}_c}{dt^2}$ **.**

The second term is $\int_V \underset{\sim}{\Gamma} \times \rho(y_i)\frac{d^2\underset{\sim}{r}_c}{dt^2}dV = \int_V \rho(y_i)\underset{\sim}{\Gamma}dV \times \frac{d^2\underset{\sim}{r}_c}{dt^2} = \underset{\sim}{0}$ **, since (** $\int_V \rho(y_i)\underset{\sim}{\Gamma}dV = \underset{\sim}{0}$ **) by definition of the CoM.**

The third term is $\int_V \underset{\sim}{r}_c \times \rho(y_i)\frac{d^2\underset{\sim}{\Gamma}}{dt^2}dV = \underset{\sim}{r}_c \times \int_V \rho(y_i)\frac{d^2\underset{\sim}{\Gamma}}{dt^2}dV = \underset{\sim}{r}_c \times \frac{d^2}{dt^2}\left[\int_V \rho(y_i)\underset{\sim}{\Gamma}dV\right] = 0$ **.**

Angular Momentum about the Center of Mass

$$\frac{d(\underset{\sim}{H}_o)}{dt} = \underset{\sim}{r}_c \times m\frac{d^2\underset{\sim}{r}_c}{dt^2} + \int_V \underset{\sim}{\Gamma} \times \rho(y_i)\frac{d^2\underset{\sim}{\Gamma}}{dt^2}dV$$

Defining the angular momentum about the center of mass as $\frac{d(\underset{\sim}{H}_c)}{dt} = \int_V \underset{\sim}{\Gamma} \times \rho(y_i)\frac{d^2\underset{\sim}{\Gamma}}{dt^2}dV$ **,**

$$\frac{d(\underset{\sim}{H}_o)}{dt} = \underset{\sim}{r}_c \times m\frac{d^2\underset{\sim}{r}_c}{dt^2} + \frac{d(\underset{\sim}{H}_c)}{dt} \quad .$$

<u>**Note that this definition is different from the definition of angular momentum about a fixed point.**</u>

Therefore, the derivative of angular momentum is: $\frac{d(\underset{\sim}{H}_o)}{dt} = \underset{\sim}{r}_c \times \underset{\sim}{F} + \frac{d(\underset{\sim}{H}_c)}{dt} = \underset{\sim}{M}_o$ **.**

From earlier, the sum of moments about the fixed point is: $\underset{\sim}{M}_o = \sum_{i=1}^{n} \underset{\sim}{r} \times \underset{\sim}{F}_i$ **. Substituting** $\underset{\sim}{r} = \underset{\sim}{r}_c + \underset{\sim}{\Gamma}$**,**

$$\underset{\sim}{M}_o = \sum_{i=1}^{n} (\underset{\sim}{r}_c + \underset{\sim}{\Gamma}) \times \underset{\sim}{F}_i = \sum_{i=1}^{n} \underset{\sim}{r}_c \times \underset{\sim}{F}_i + \sum_{i=1}^{n} \underset{\sim}{\Gamma} \times \underset{\sim}{F}_i = \underset{\sim}{r}_c \times \sum_{i=1}^{n} \underset{\sim}{F}_i + \sum_{i=1}^{n} \underset{\sim}{\Gamma} \times \underset{\sim}{F}_i = \underset{\sim}{r}_c \times \underset{\sim}{F} + \underset{\sim}{M}_c \quad .$$

Conservation of Angular Momentum about Center of Mass

Since $\frac{d(\underset{\sim}{H}_o)}{dt} = \underset{\sim}{r}_c \times \underset{\sim}{F} + \frac{d(\underset{\sim}{H}_c)}{dt} = \underset{\sim}{M}_o = \underset{\sim}{r}_c \times \underset{\sim}{F} + \underset{\sim}{M}_c$ **,**

$$\frac{d(\underset{\sim}{H}_c)}{dt} = \underset{\sim}{M}_c \, .$$

The time rate of change of "angular momentum" with respect to the center of mass is equal to the sum of the moments of the external forces about the center of mass.

Remember that $\frac{d(\underset{\sim}{H}_c)}{dt} = \int_V \underset{\sim}{\Gamma} \times \rho(y_i) \frac{d^2 \underset{\sim}{\Gamma}}{dt^2} dV$ **and** $\underset{\sim}{M}_c = \sum_{i=1}^{n} \underset{\sim}{\Gamma} \times \underset{\sim}{F}_i$ **where** $\underset{\sim}{\Gamma} = y_i \underset{\sim}{e}_i$ **.**

Angular Momentum

The angular momentum about the center of mass for a rigid body is:

$$\underset{\sim}{H}_c = \int_V \left(\rho \underset{\sim}{\Gamma} \times \frac{d(\underset{\sim}{\Gamma})}{dt}\right) dV = \int_V \rho \underset{\sim}{\Gamma} \times (\underset{\sim}{\omega} \times \underset{\sim}{\Gamma}) dV \qquad \textbf{(1)}$$

where $\underset{\sim}{\omega}$ **is the absolute angular velocity for the reference frame. Note that, if** $\underset{\sim}{\Gamma} = y_i \underset{\sim}{e}_i$ **, then** $\frac{dy_i}{dt} = 0$ **by definition of the reference frame.**

Let $\underset{\sim}{e}_i$ **be a Cartesian reference frame, such that** $dV = dy_1 dy_2 dy_3$ **and let** $\underset{\sim}{\omega} = \omega_i \underset{\sim}{e}_i$ **. Therefore,**

$$\underset{\sim}{\Gamma} \times (\underset{\sim}{\omega} \times \underset{\sim}{\Gamma}) = e_{ijk} y_j e_{kmn} \omega_m y_n \underset{\sim}{e}_i = e_{kij} e_{kmn} y_j \omega_m y_n \underset{\sim}{e}_i = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(y_j y_n)(\omega_m \underset{\sim}{e}_i) \quad . \qquad \textbf{(2)}$$

Since $\underset{\sim}{e}_i$ **and** $\underset{\sim}{\omega}$ **are constant <u>with respect to the integration</u>,** $\underset{\sim}{H}_c = \underset{\sim}{e}_i \left\{ \int_V \rho (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(y_j y_n) dV \right\} \omega_m$

Moment of Inertia

Inserting a Kronecker delta between the integral and the angular velocity yields:

$$\underset{\sim}{H}_c = \underset{\sim}{e}_i \left\{ \int_V \rho(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(y_j y_n) dV \right\} \delta_{mk}\omega_k = \underset{\sim}{e}_i \left\{ \int_V \rho(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(y_j y_n) dV \right\} (\underset{\sim}{e}_m \bullet \underset{\sim}{e}_k)\omega_k \quad . \tag{1}$$

Defining the Moment of Inertia tensor as: $\underset{\sim}{I}_c = \underset{\sim}{e}_i (I_{im}) \underset{\sim}{e}_m$ **, where** $I_{im} = \int_V \rho(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(y_j y_n) dV$,

it can be seen that: $\underset{\sim}{H}_c = \underset{\sim}{I}_c \bullet \underset{\sim}{\omega}$ **. For computation purposes, the components of the moment of inertia tensor are:**

$$\begin{aligned}
I_{11} &= \int_V \rho(\delta_{11}\delta_{jn} - \delta_{1n}\delta_{1m})(y_j y_n) dV = \int_V \rho(y_j y_j - y_1^2) dV = \int_V \rho(y_2^2 + y_3^2) dV \\
I_{22} &= \int_V \rho(\delta_{22}\delta_{jn} - \delta_{2n}\delta_{2m})(y_j y_n) dV = \int_V \rho(y_j y_j - y_2^2) dV = \int_V \rho(y_1^2 + y_3^2) dV \\
I_{33} &= \int_V \rho(\delta_{33}\delta_{jn} - \delta_{3n}\delta_{3m})(y_j y_n) dV = \int_V \rho(y_j y_j - y_3^2) dV = \int_V \rho(y_1^2 + y_2^2) dV \\
I_{12} &= \int_V \rho(\delta_{12}\delta_{jn} - \delta_{1n}\delta_{2m})(y_j y_n) dV = \int_V \rho(-y_1 y_2) dV = I_{21} \\
I_{23} &= \int_V \rho(\delta_{23}\delta_{jn} - \delta_{2n}\delta_{3m})(y_j y_n) dV = \int_V \rho(-y_2 y_3) dV = I_{32} \\
I_{31} &= \int_V \rho(\delta_{31}\delta_{jn} - \delta_{3n}\delta_{1m})(y_j y_n) dV = \int_V \rho(-y_3 y_1) dV = I_{13}
\end{aligned} \tag{2}$$

Derivative of Angular Momentum

To use Angular Momentum equations, the derivative of the angular momentum is required.

First, consider the angular momentum about the center of mass, $\underset{\sim}{H}_c = \underset{\sim}{I}_c \bullet \underset{\sim}{\omega}$ **.**

$$\dot{\underset{\sim}{H}}_c = \frac{d(\underset{\sim}{I}_c \bullet \underset{\sim}{\omega})}{dt} = \frac{d(\underset{\sim}{I}_c)}{dt} \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_c \bullet \frac{d(\underset{\sim}{\omega})}{dt}$$

Note that the dot product between the moment of inertia tensor and the angular velocity is <u>not</u> commutative.

First, evaluate the derivative of the angular momentum:

$$\frac{d(\underset{\sim}{\omega})}{dt} = \frac{d(\omega_i \underset{\sim}{e}_i)}{dt} = \frac{d(\omega_i)}{dt}\underset{\sim}{e}_i + \omega_i \frac{d(\underset{\sim}{e}_i)}{dt} = \alpha_i \underset{\sim}{e}_i + \underset{\sim}{\omega} \times \underset{\sim}{\omega} = \alpha_i \underset{\sim}{e}_i \equiv \underset{\sim}{\alpha} \quad .$$

The quantity $\underset{\sim}{\alpha}$ **is known as the angular acceleration.**

Now consider $\frac{d(\underset{\sim}{I}_c)}{dt} = \frac{d(\underset{\sim}{e}_i I_{im} \underset{\sim}{e}_m)}{dt} = \frac{d(\underset{\sim}{e}_i)}{dt} I_{im} \underset{\sim}{e}_m + \underset{\sim}{e}_i \frac{d(I_{im})}{dt} \underset{\sim}{e}_m + \underset{\sim}{e}_i I_{im} \frac{d(\underset{\sim}{e}_m)}{dt}$**.**

Remembering, from the definition of the moment of inertia tensor, the quantities, I_{im}**, are constants.**

Therefore, $\frac{d(\underset{\sim}{I}_c)}{dt} = \underset{\sim}{\omega} \times \underset{\sim}{I}_c + \underset{\sim}{I}_c \times \underset{\sim}{\omega}$ **and** $\dot{\underset{\sim}{H}}_c = (\underset{\sim}{\omega} \times \underset{\sim}{I}_c + \underset{\sim}{I}_c \times \underset{\sim}{\omega}) \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_c \bullet \underset{\sim}{\alpha}$ **.**

Simplifying the Derivative of the Moment of Inertia

The term $(\underset{\sim}{I}_c \times \underset{\sim}{\omega}) \bullet \underset{\sim}{\omega}$ can be simplified, using indicial notation,

$$(\underset{\sim}{I}_c \times \underset{\sim}{\omega}) \bullet \underset{\sim}{\omega} = ((\underset{\sim}{e}_j I_{jm} \underset{\sim}{e}_m) \times (\omega_i \underset{\sim}{e}_i)) \bullet (\omega_n \underset{\sim}{e}_n) = (\underset{\sim}{e}_j I_{jm} e_{mik} \omega_i \underset{\sim}{e}_k) \bullet (\omega_n \underset{\sim}{e}_n) = \underset{\sim}{e}_j I_{jm} e_{mik} \omega_i \omega_n \delta_{kn} = \underset{\sim}{e}_j I_{jm} e_{mik} \omega_i \omega_k ,$$

This term has an anti-symmetric quantity (e_{mik}) summed with a symmetric quantity ($\omega_i \omega_k$), which is zero.

Returning to invariant notation,

$$\dot{\underset{\sim}{H}}_c = (\underset{\sim}{\omega} \times \underset{\sim}{I}_c) \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_c \bullet \underset{\sim}{\alpha} .$$

Parentheses are unnecessary in the first term, since $\underset{\sim}{I}_c$ is a second rank tensor. The cross product on the left operates on $\underset{\sim}{I}_c$ ‘s left unit vector, while the dot product on the right operates on $\underset{\sim}{I}_c$ ‘s right unit vector.

Angular Momentum about the Center of Mass

For any system of particles, it was shown that the angular momentum about the center of mass is related to the sum of the moments of external forces about the center of mass by

$$\dot{\underset{\sim}{H}}_c = \underset{\sim}{M}_c \text{ .}$$

Since a rigid body is a system of particles, this equation is valid for any rigid body. From the previous slides,

$$\dot{\underset{\sim}{H}}_c = \underset{\sim}{\omega} \times \underset{\sim}{I}_c \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_c \bullet \underset{\sim}{\alpha}$$

Hence,

$$\underset{\sim}{M}_c = \underset{\sim}{\omega} \times \underset{\sim}{I}_c \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_c \bullet \underset{\sim}{\alpha} \quad ,$$

which can be rewritten in components as:

x component $M_x = I_{xx}\dot{\omega}_x + I_{xy}(\dot{\omega}_y - \omega_x\omega_z) + I_{xz}(\dot{\omega}_z + \omega_x\omega_y) + (I_{zz} - I_{yy})(\omega_y\omega_z) + I_{yz}(\omega_y^2 - \omega_z^2)$

y component $M_y = I_{yy}\dot{\omega}_y + I_{yz}(\dot{\omega}_z - \omega_x\omega_y) + I_{xy}(\dot{\omega}_x + \omega_y\omega_z) + (I_{xx} - I_{zz})(\omega_x\omega_z) + I_{xz}(\omega_z^2 - \omega_x^2)$

z component $M_z = I_{zz}\dot{\omega}_z + I_{xz}(\dot{\omega}_x - \omega_y\omega_z) + I_{yz}(\dot{\omega}_y + \omega_x\omega_z) + (I_{yy} - I_{xx})(\omega_x\omega_y) + I_{xy}(\omega_x^2 - \omega_y^2)$

Some Linear Algebra

The moment of inertia was seen to be a second rank tensor, whose components are given by $I_{im} = \int_V \rho(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(y_j y_n)dV$ **, where** $\rho(y_i)$ **is the density of the body and** y_i **are the components of the position vector of each mass element, relative to the center of mass.**

The components of a second rank tensor can be arranged into a three-by-three matrix, which, in this case, is symmetric.

By the definition of the components, the symmetric matrix will be non-singular and positive definite. Hence, it will have three non-zero real eigenvalues.

If the eigenvalues are distinct, there will be three eigenvectors associated with those eigenvalues, $\underset{\sim}{\chi}_1, \underset{\sim}{\chi}_2, \underset{\sim}{\chi}_3$**, which can be normalized to unit magnitude. Let the components of these normalized eigenvectors be** $\chi_{i1}, \chi_{i2}, \chi_{i3}$**. A matrix can be formed from these eigenvectors,** χ_{ji}**. This matrix is guaranteed, by the definition of a normalized eigenvector, to be orthonormal, just like a coordinate transformation matrix. If the moment of inertia tensor is pre- and post- multiplied by this matrix, such that** $I^P_{jk} = \chi^{-1}_{ji} I_{im} \chi_{mk}$**, then the new matrix,** I^P_{im}**, will be diagonalized. (This is a well known theorem from linear algebra.)**

If two eigenvalues are distinct, the third vector to use in the transformation is obtained by crossing the two eigenvectors.

If one eigenvalue is distinct, then a coordinate transformation to make the eigenvector, $\underset{\sim}{e}_1$**, can be done. The other two vectors are** $\underset{\sim}{e}_2$ **and** $\underset{\sim}{e}_3$**.**

Principal Moments of Inertia

Since χ_{ji} is orthonormal and I_{im} is symmetric, $\chi_{ji}^{-1} = \chi_{ji}^{T}$, which is a property of a coordinate transformation matrix. Therefore, $I_{jk}^{P} = \chi_{ji}^{T} I_{im} \chi_{mk}$.

Inverting this gives us $\chi_{ij} I_{jk}^{P} \chi_{km}^{T} = I_{im}$.

By identifying a coordinate transformation with χ_{ji}, the moment of inertia tensor (which is an invariant) is

$$\underset{\sim}{I} = \underset{\sim}{e}_i (I_{im}) \underset{\sim}{e}_m = \underset{\sim}{e}_i (\chi_{ij} I_{jk}^{P} \chi_{km}^{T}) \underset{\sim}{e}_m = \underset{\sim}{e}'_j (I_{jk}^{P}) \underset{\sim}{e}'_k$$

The coordinate transformation matrix between the $\underset{\sim}{e}_i$ system and the $\underset{\sim}{e}'_i$ system is $a_{ji} = \chi_{ji}^{T} = \underset{\sim}{e}'_j \bullet \underset{\sim}{e}_i$.

The diagonal values in the moment of inertia tensor are called the principal moments of inertia.

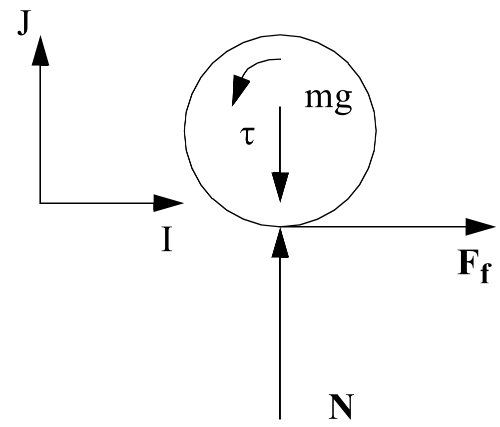
$$I_{jk}^{P} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} = J_{(j)} \delta_{jk} \quad \textbf{and} \quad \begin{array}{ll} \text{x component} & M_x = J_x \dot{\omega}_x + (J_z - J_y)(\omega_y \omega_z) \\ \text{y component} & M_y = J_y \dot{\omega}_y + (J_x - J_z)(\omega_x \omega_z) \\ \text{z component} & M_z = J_z \dot{\omega}_z + (J_y - J_x)(\omega_x \omega_y) \end{array} .$$

The right hand equations are called Euler's Equations and can save considerable time in making computations.

Example: Wheel Rolling in a Line

The forces acting on the wheel are:

$$\underset{\sim}{F}_G = N\underset{\sim}{J} + F_f\underset{\sim}{I}$$
$$\underset{\sim}{W} = -mg\underset{\sim}{J}\ .$$
$$\underset{\sim}{\tau} = \tau\underset{\sim}{K}$$



FBD of Wheel

A vector pointing from the wheel COM to the point of contact is: $\underset{\sim}{r} = -r\underset{\sim}{J}$.

The acceleration of the center of mass of the wheel is $\underset{\sim}{a} = \ddot{x}\underset{\sim}{I}$.

The angular velocity of a coordinate system rotating with the wheel is $\underset{\sim}{\omega} = \dot{\theta}\underset{\sim}{k}$.

The moment of inertia tensor in body fixed coordinates is: $\underset{\sim}{I} = I_w(\underset{\sim}{i}\underset{\sim}{i} + \underset{\sim}{j}\underset{\sim}{j}) + J_w\underset{\sim}{k}\underset{\sim}{k}$.

The angular momentum about the center of mass is: $\underset{\sim}{H}_{cm} = \underset{\sim}{I} \bullet \underset{\sim}{\omega} = [I_w(\underset{\sim}{i}\underset{\sim}{i} + \underset{\sim}{j}\underset{\sim}{j}) + J_w\underset{\sim}{k}\underset{\sim}{k}] \bullet \dot{\theta}\underset{\sim}{k} = J_w\dot{\theta}\underset{\sim}{k} = J_w\dot{\theta}\underset{\sim}{K}$. **The derivative of this yields:** $\dot{\underset{\sim}{H}}_{cm} = J_w\ddot{\theta}\underset{\sim}{K}$.

Summing forces yields: $\underset{\sim}{F}_G + \underset{\sim}{W} = N\underset{\sim}{J} + F_f\underset{\sim}{I} - mg\underset{\sim}{J} = m\underset{\sim}{a} = m\ddot{x}\underset{\sim}{I}$.

This yields the Equation of Motion: $F_f = m\ddot{x}$ **and the Equation of Contraint:** $N = mg$.

Wheel Moving in a Line

Summing moments about the center of mass yields: $\underset{\sim}{r} \times \underset{\sim}{F}_G + \underset{\sim}{\tau} = (-r\underset{\sim}{J}) \times (N\underset{\sim}{J} + F_f\underset{\sim}{I}) + \tau\underset{\sim}{K} = (rF_f + \tau)\underset{\sim}{K} = J_w\ddot{\theta}\underset{\sim}{K}$.

This tells us the second Equation of Motion: $(rF_f + \tau) = J_w\ddot{\theta}$.

The no slip constraint is $\dot{x} + r\dot{\theta} = 0$.

Assembling the two equations of motion in matrix form gives the equation: $\begin{bmatrix} J_w & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} rF_f \\ F_f \end{bmatrix} + \begin{bmatrix} \tau \\ 0 \end{bmatrix}$.

Taking the derivative of the no slip constraint and putting it into matrix form gives: $\begin{bmatrix} r & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = 0$.

The equations of motion can be augmented with this additional row to be: $\begin{bmatrix} J_w & 0 & -r \\ 0 & m & -1 \\ r & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \\ F_f \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \\ 0 \end{bmatrix}$.

This set of equations could be simulated to yield both the motion of the wheel under no slip conditions and the value of the no slip constraint force. Through row reduction, the value for both state variables (which are not independent) could be deduced.

Wheel Moving in a Line

Defining $[A] = \begin{bmatrix} J_w & 0 \\ 0 & m \end{bmatrix}$, $[c] = \begin{bmatrix} r \\ 1 \end{bmatrix}$, $[\ddot{\xi}] = \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix}$, **and** $[u] = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$, **the EoM and the Constraint can be written as:**

$$[A][\ddot{\xi}] = F_f[c] + [u], \quad [c]^T[\ddot{\xi}] = 0. \qquad \textbf{(1)}$$

The matrix $[A]$ **will be positive definite and therefore invertible. Hence,** $[\ddot{\xi}] = F_f[A]^{-1}[c] + [A]^{-1}[u]$.

Using the constraint equation gives: $[c]^T[\ddot{\xi}] = F_f[c]^T[A]^{-1}[c] + [c]^T[A]^{-1}[u] = 0$.

This can be solved to yield the value for the friction force: $F_f = -\dfrac{[c]^T[A]^{-1}[u]}{[c]^T[A]^{-1}[c]}$.

The real friction force is: $(F_f)_{actual} = \begin{cases} F_f & \text{otherwise} \\ \mu_k N\left(\dfrac{F_f}{|F_f|}\right) & |F_f| > \mu_s N \quad \text{or } \dot{x} + r\dot{\theta} \neq 0 \end{cases}$. **The true EoM is:**

$$\begin{bmatrix} J_w & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} r \\ 1 \end{bmatrix}(F_f)_{actual} + \begin{bmatrix} \tau \\ 0 \end{bmatrix}.$$

State Space of a Wheel Moving in a Line

The EoM of a wheel moving in a line is $\begin{bmatrix} J_w & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} r \\ 1 \end{bmatrix} (F_f)_{actual} + \begin{bmatrix} \tau \\ 0 \end{bmatrix}$**.**

The friction force is determined from $(F_f)_{actual} = \begin{cases} F_f & |F_f| \le \mu_s N \\ \mu_k N\left(\dfrac{F_f}{|F_f|}\right) & |F_f| > \mu_s N \end{cases}$ **where** $F_f = -\dfrac{\begin{bmatrix} r & 1 \end{bmatrix} \begin{bmatrix} J_w & 0 \\ 0 & m \end{bmatrix}^{-1} \begin{bmatrix} \tau \\ 0 \end{bmatrix}}{\begin{bmatrix} r & 1 \end{bmatrix} \begin{bmatrix} J_w & 0 \\ 0 & m \end{bmatrix}^{-1} \begin{bmatrix} r \\ 1 \end{bmatrix}}$**.**

Define the state space vector as $[y] = \begin{bmatrix} \theta \\ x \\ \dot{\theta} \\ \dot{x} \end{bmatrix}$**. The state space equation becomes:**

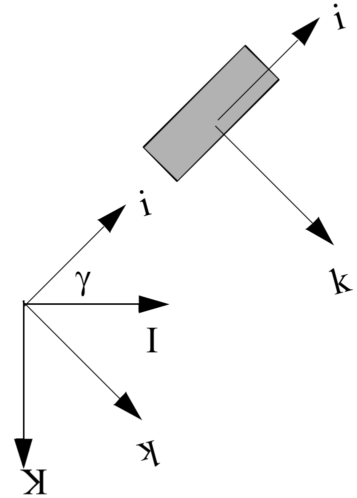
$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} J_w & 0 \\ 0 & m \end{bmatrix} \end{bmatrix} [\dot{y}] = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} [y] + \begin{bmatrix} 0 \\ 0 \\ r \\ 1 \end{bmatrix} (F_f)_{actual} + \begin{bmatrix} 0 \\ 0 \\ \tau \\ 0 \end{bmatrix}$$

Example: Wheel Rolling in a Plane

The forces acting on the wheel are:

$$\mathbf{F}_G = N\mathbf{j} + F_{fx}\mathbf{i} + F_{fz}\mathbf{k}$$
$$\mathbf{W} = -mg\mathbf{J}$$
$$\boldsymbol{\tau} = \tau\mathbf{k} + \lambda\mathbf{j} + \mu\mathbf{i}$$

Overview of Wheel



A vector pointing from the wheel COM to the point of contact is: $\mathbf{r} = -r\mathbf{j}$.

The acceleration of the center of mass of the wheel is $\mathbf{a} = \ddot{x}\mathbf{I} + \ddot{z}\mathbf{K}$.

The angular velocity of the coordinate system $\mathbf{i}, \mathbf{j}, \mathbf{k}$ **moving with the wheel (but not rotating) is** $\boldsymbol{\omega} = \dot{\gamma}\mathbf{j}$.

The angular velocity of a coordinate system $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ **rotating with the wheel is** $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}' + \dot{\gamma}\mathbf{j} = \dot{\theta}\mathbf{k}' + \dot{\gamma}(\cos\theta\,\mathbf{i}' + \sin\theta\,\mathbf{j}')$.

The moment of inertia tensor in body fixed coordinates is: $\mathbf{I} = I_w(\mathbf{i}'\mathbf{i}' + \mathbf{j}'\mathbf{j}') + J_w\mathbf{k}'\mathbf{k}'$.

The angular momentum about the center of mass is:

$$\mathbf{H}_{cm} = \mathbf{I}\bullet\boldsymbol{\omega} = [I_w(\mathbf{i}'\mathbf{i}' + \mathbf{j}'\mathbf{j}') + J_w\mathbf{k}'\mathbf{k}']\bullet[\dot{\theta}\mathbf{k}' + \dot{\gamma}(\cos\theta\,\mathbf{i}' + \sin\theta\,\mathbf{j}')] = I_w\dot{\gamma}(\cos\theta\,\mathbf{i}' + \sin\theta\,\mathbf{j}') + J_w\dot{\theta}\mathbf{k}' = I_w\dot{\gamma}\mathbf{j} + J_w\dot{\theta}\mathbf{k}.$$

The derivative of this yields: $\dot{\mathbf{H}}_{cm} = I_w\ddot{\gamma}\mathbf{j} + \ddot{J}_w\ddot{\theta}\mathbf{k} + (\dot{\gamma}\mathbf{j})\times(I_w\dot{\gamma}\mathbf{j} + J_w\dot{\theta}\mathbf{k}) = I_w\ddot{\gamma}\mathbf{j} + J_w\ddot{\theta}\mathbf{k} + J_w\dot{\theta}\dot{\gamma}\mathbf{i}$.

Example: Wheel Rolling in a Plane

Summing forces yields: $\underset{\sim}{F}_G + \underset{\sim}{W} = N\underset{\sim}{j} + F_{fx}\underset{\sim}{i} + F_{fz}\underset{\sim}{k} - mg\underset{\sim}{J} = m\underset{\sim}{a} = m(\ddot{x}\underset{\sim}{I} + \ddot{z}\underset{\sim}{K})$.

Putting everything into the coordinate system that is moving with the wheel (but not rotating) yields:

$$N\underset{\sim}{j} + F_{fx}\underset{\sim}{i} + F_{fz}\underset{\sim}{k} - mg\underset{\sim}{j} = m[(\ddot{x}\cos\gamma - \ddot{z}\sin\gamma)\underset{\sim}{i} + (\ddot{x}\sin\gamma + \ddot{z}\cos\gamma)\underset{\sim}{k}]$$

This yields the Equations of Motion: $\begin{aligned} F_{fx} &= m(\ddot{x}\cos\gamma - \ddot{z}\sin\gamma) \\ F_{fz} &= m(\ddot{x}\sin\gamma + \ddot{z}\cos\gamma) \end{aligned}$ **and the Equation of Contraint:** $N = mg$.

The EoM can be rewritten as $\begin{aligned} m\ddot{x} &= F_{fx}\cos\gamma + F_{fz}\sin\gamma \\ m\ddot{z} &= F_{fz}\cos\gamma - F_{fx}\sin\gamma \end{aligned}$.

Summing moments about the center of mass yields:

$$\underset{\sim}{r} \times \underset{\sim}{F}_G + \underset{\sim}{\tau} = (-r\underset{\sim}{j}) \times (N\underset{\sim}{j} + F_{fx}\underset{\sim}{i} + F_{fz}\underset{\sim}{k}) + \tau\underset{\sim}{k} = r(F_{fx}\underset{\sim}{k} - F_{fz}\underset{\sim}{i}) + \tau\underset{\sim}{k} + \lambda\underset{\sim}{j} + \mu\underset{\sim}{i} = I_w\ddot{\gamma}\underset{\sim}{j} + J_w\ddot{\theta}\underset{\sim}{k} + J_w\dot{\theta}\dot{\gamma}\underset{\sim}{i}.$$

This tells us the third and fourth Equations of Motion: $\begin{aligned} rF_{fx} + \tau &= J_w\ddot{\theta} \\ \lambda &= I_w\ddot{\gamma} \end{aligned}$ **and an extra constraint,** $J_w\dot{\theta}\dot{\gamma} + rF_{fz} = \mu$.

Example: Wheel Rolling in a Plane

Collecting these equations into matrix notation:

$$\begin{bmatrix} J_w & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \\ \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} F_{fx} \\ F_{fz} \end{bmatrix} + \begin{bmatrix} \tau \\ \lambda \\ 0 \\ 0 \end{bmatrix}$$

The no slip conditions are: $\begin{aligned} \dot{x}\cos\gamma + \dot{z}\sin\gamma + r\dot{\theta} &= 0 \\ -\dot{x}\sin\gamma + \dot{z}\cos\gamma &= 0 \end{aligned}$.

Taking derivatives of this condition yields: $\begin{aligned} \ddot{x}\cos\gamma + \ddot{z}\sin\gamma + r\ddot{\theta} &= \dot{\gamma}(\dot{x}\sin\gamma - \dot{z}\cos\gamma) \\ -\ddot{x}\sin\gamma + \ddot{z}\cos\gamma &= \dot{\gamma}(\dot{x}\cos\gamma + \dot{z}\sin\gamma) \end{aligned}$.

In matrix form:

$$\begin{bmatrix} r & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \\ \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dot{\gamma}(\dot{x}\sin\gamma - \dot{z}\cos\gamma) \\ \dot{\gamma}(\dot{x}\cos\gamma + \dot{z}\sin\gamma) \end{bmatrix}$$

Example: Wheel Rolling in a Plane

Defining $[A] = \begin{bmatrix} J_w & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$, $[c] = \begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$, $[\ddot{\xi}] = \begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \\ \ddot{x} \\ \ddot{z} \end{bmatrix}$, **and** $[u] = \begin{bmatrix} \tau \\ \lambda \\ 0 \\ 0 \end{bmatrix}$, $[f] = \begin{bmatrix} F_{fx} \\ F_{fz} \end{bmatrix}$, $[g] = \begin{bmatrix} \dot{\gamma}(\dot{x}\sin\gamma - \dot{z}\cos\gamma) \\ \dot{\gamma}(\dot{x}\cos\gamma + \dot{z}\sin\gamma) \end{bmatrix}$ **the EoM and the Constraint can be written as:**

$$[A][\ddot{\xi}] = [c][f] + [u] \text{, } [c]^T[\ddot{\xi}] = [g]. \qquad \textbf{(1)}$$

The matrix $[A]$ **will be positive definite and therefore invertible. Hence,** $[\ddot{\xi}] = [A]^{-1}[c][f] + [A]^{-1}[u]$.

Using the constraint equation gives:

$$[c]^T[\ddot{x}] = [c]^T[A]^{-1}[c][f] + [c]^T[A]^{-1}[u] = [g]. \qquad \textbf{(2)}$$

This can be solved to yield the value for the friction force:

$$[f] = \left([c]^T[A]^{-1}[c]\right)^{-1}\left([g] - [c]^T[A]^{-1}[u]\right). \qquad \textbf{(3)}$$

Example: Wheel Rolling in a Plane

The real friction forces are:

$$(f_{fx})_{actual} = \begin{cases} f_{fx} & \text{otherwise} \\ \mu_{kx} N \,\mathrm{sgn}(f_{fx}) & |f_{fx}| > \mu_{sx} N \quad \text{or} \quad \dot{x}\cos\gamma + \dot{z}\sin\gamma + r\dot{\theta} \neq 0 \end{cases}$$

$$(f_{fz})_{actual} = \begin{cases} f_{fz} & \text{otherwise} \\ \mu_{kz} N \,\mathrm{sgn}(f_{fz}) & |f_{fz}| > \mu_{sz} N \quad \text{or} \quad -\dot{x}\sin\gamma + \dot{z}\cos\gamma \neq 0 \end{cases}$$

The true EoM is: $\begin{bmatrix} J_w & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \\ \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} [f]_{actual} + \begin{bmatrix} \tau \\ \lambda \\ 0 \\ 0 \end{bmatrix}$.

Note that the coefficients of friction admit that the wheel may have different coefficients to resist side slip and to maintain rolling. Further, in the event of slippage, the resulting forces may again be different.

This reflects that wheel treads may be designed to include these kinds of effects.

Also note that one component of the friction could be in slip and the other in no slip simultaneously.

State Space of a Wheel Moving in a Plane

The EoM of a wheel moving in a plane is $$\begin{bmatrix} J_w & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \\ \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} F_{fx} \\ F_{fz} \end{bmatrix} + \begin{bmatrix} \tau \\ \lambda \\ 0 \\ 0 \end{bmatrix}$$ **.**

The friction force is determined from

$$(f_{fx})_{actual} = \begin{cases} f_{fx} & \text{otherwise} \\ \mu_{kx} N \operatorname{sgn}(f_{fx}) & |f_{fx}| > \mu_{sx} N \quad \text{or} \quad \dot{x}\cos\gamma + \dot{z}\sin\gamma + r\dot{\theta} \neq 0 \end{cases}$$

$$(f_{fz})_{actual} = \begin{cases} f_{fz} & \text{otherwise} \\ \mu_{kz} N \operatorname{sgn}(f_{fz}) & |f_{fz}| > \mu_{sz} N \quad \text{or} \quad -\dot{x}\sin\gamma + \dot{z}\cos\gamma \neq 0 \end{cases}$$

where

$$[f] = \left(\begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}^T \begin{bmatrix} J_w & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}^{-1} \begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} \dot{\gamma}(\dot{x}\sin\gamma - \dot{z}\cos\gamma) \\ \dot{\gamma}(\dot{x}\cos\gamma + \dot{z}\sin\gamma) \end{bmatrix} - \begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}^T \begin{bmatrix} J_w & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}^{-1} \begin{bmatrix} \tau \\ \lambda \\ 0 \\ 0 \end{bmatrix} \right)$$

State Space of a Wheel Moving in a Plane

Define the state space vector as $[y] = \begin{bmatrix} \theta \\ \gamma \\ x \\ z \\ \dot{\theta} \\ \dot{\gamma} \\ \dot{x} \\ \dot{z} \end{bmatrix}$**. The state space equation becomes:**

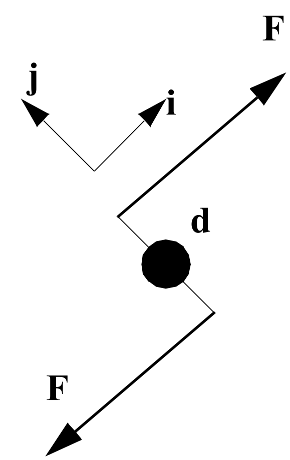
$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} J_w & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \end{bmatrix} [\dot{y}] = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} [y] + \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} r & 0 \\ 0 & 0 \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \end{bmatrix} [f]_{actual} + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \tau \\ \lambda \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

Couples

Occasionally a situation presents itself when a "pure torque" is imposed on a system. This usually occurs when motors or moment contraint forces arise.

There is no such thing as a pure torque!

Consider instead a "couple" of forces, one the same magnitude but oppositely directed as the other and acting some distance apart.



These forces are external and create a moment $\underset{\sim}{M} = \left(\frac{d}{2}\underset{\sim}{j}\right) \times (F\underset{\sim}{i}) + \left(-\frac{d}{2}\underset{\sim}{j}\right) \times (-F\underset{\sim}{i}) = Fd\underset{\sim}{k}$**.**

However, in the sum of the forces, $\underset{\sim}{F} - \underset{\sim}{F} = \underset{\sim}{0}$**.**

Even though there is no such thing as a pure moment, a couple of forces acting at a distance apart create the effect of a pure torque.

In the most likely case, motor torques, the interaction between the magnetic fields in the rotor and stator create forces acting at the outside of the rotor. Motors are designed so that these forces are equal and oppositely directed. They act as a pure torque on the output shaft.

Rigid Body Rotating About a Fixed Point

Consider a rigid body which is constrained to rotate about a fixed point such that the components for the center of mass vector do not change when written in the body fixed coordinate system.

The angular momentum for this rigid body is $\underset{\sim}{H}_o = \int_V \rho \underset{\sim}{r} \times \frac{d\underset{\sim}{r}}{dt} dV$**. Remembering that** $\underset{\sim}{r} = \underset{\sim}{r}_c + \underset{\sim}{\Gamma}$**, this becomes:**

$$\underset{\sim}{H}_o = \int_V \rho (\underset{\sim}{r}_c + \underset{\sim}{\Gamma}) \times \frac{d(\underset{\sim}{r}_c + \underset{\sim}{\Gamma})}{dt} dV = \int_V \rho \underset{\sim}{r}_c \times \frac{d\underset{\sim}{r}_c}{dt} dV + \int_V \rho \underset{\sim}{\Gamma} \times \frac{d\underset{\sim}{\Gamma}}{dt} dV .$$

The cross terms were eliminated by the definition of the center of mass: $\int_V \rho \underset{\sim}{\Gamma} dV = 0$.

Since the components of the points in the body relative to the center of mass are fixed in the body fixed coordinate system and since the same is true for this situation's center of mass vector,

$$\underset{\sim}{H}_o = m\underset{\sim}{r}_c \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_c) + \int_V \rho \underset{\sim}{\Gamma} \times (\underset{\sim}{\omega} \times \underset{\sim}{\Gamma}) dV .$$

From the definition of the moment of inertia:

$$\underset{\sim}{H}_o = m\underset{\sim}{r}_c \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_c) + \underset{\sim}{I}_c \bullet \underset{\sim}{\omega} .$$

Parallel Axis Theorem

Recall, from vector calculus, that $\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = (\underset{\sim}{A} \bullet \underset{\sim}{C})\underset{\sim}{B} - (\underset{\sim}{A} \bullet \underset{\sim}{B})\underset{\sim}{C}$**. This can be manipulated into a more complicated (but more useful) tensor equation, by defining the identity tensor,** $\underset{\sim}{\Delta} = \underset{\sim}{e}_i \delta_{ij} \underset{\sim}{e}_j$ **,**
$\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = (\underset{\sim}{A} \bullet \underset{\sim}{C})(\underset{\sim}{\Delta} \bullet \underset{\sim}{B}) - ((\underset{\sim}{C}\underset{\sim}{A}) \bullet \underset{\sim}{B}) = ((\underset{\sim}{C} \bullet \underset{\sim}{A})\underset{\sim}{\Delta} - \underset{\sim}{C}\underset{\sim}{A}) \bullet \underset{\sim}{B}$ **.**

Then, $\underset{\sim}{r}_c \times m(\underset{\sim}{\omega} \times \underset{\sim}{r}_c) = m[(\underset{\sim}{r}_c \bullet \underset{\sim}{r}_c)\underset{\sim}{\Delta} - \underset{\sim}{r}_c\underset{\sim}{r}_c] \bullet \underset{\sim}{\omega}$ **and** $\frac{d}{dt}\{[\underset{\sim}{I}_c + m((\underset{\sim}{r}_c \bullet \underset{\sim}{r}_c)\underset{\sim}{\Delta} - \underset{\sim}{r}_c\underset{\sim}{r}_c)] \bullet \underset{\sim}{\omega}\} = \underset{\sim}{M}_o$ **.**

The parallel moment of inertia is defined as $\underset{\sim}{I}_o = \underset{\sim}{I}_c + m((\underset{\sim}{r}_c \bullet \underset{\sim}{r}_c)\underset{\sim}{\Delta} - \underset{\sim}{r}_c\underset{\sim}{r}_c)$ **and**

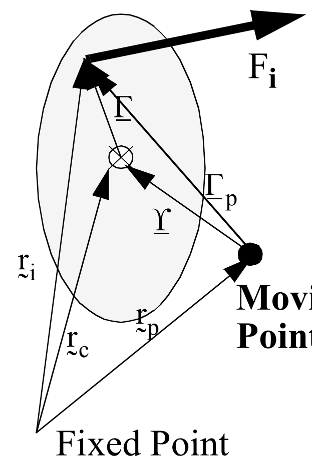
$$\frac{d}{dt}(\underset{\sim}{I}_o \bullet \underset{\sim}{\omega}) = \underset{\sim}{M}_o \ .$$

Since the components of $m((\underset{\sim}{r}_c \bullet \underset{\sim}{r}_c)\underset{\sim}{\Delta} - \underset{\sim}{r}_c\underset{\sim}{r}_c)$ **do not vary with time, the earlier work on the moment of inertia is valid and the equations of motion for a rigid body rotating about a fixed point are**

$$\underset{\sim}{\omega} \times \underset{\sim}{I}_o \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_o \bullet \underset{\sim}{\alpha} = \underset{\sim}{M}_o \ .$$

Rigid Body Rotating about a Special Moving Point

Consider a body which is rotating about a moving point, where the moving point moves with the rigid body. This point, which is not necessarily on the rigid body would appear to be a part of the rigid body. The description of the moving point is $\underset{\sim}{r}_p = \underset{\sim}{r}_c - \underset{\sim}{\Upsilon}$ **and the location of external forces relative to the center of mass is** $\underset{\sim}{\Gamma} = \underset{\sim}{\Gamma}_p - \underset{\sim}{\Upsilon}$**.**



The moment of external forces about the COM can be rewritten as

$$\underset{\sim}{M}_c = \sum_{i=1}^{n} \underset{\sim}{\Gamma} \times \underset{\sim}{F}_i = \sum_{i=1}^{n} (\underset{\sim}{\Gamma}_p - \underset{\sim}{\Upsilon}) \times \underset{\sim}{F}_i = \sum_{i=1}^{n} \underset{\sim}{\Gamma}_p \times \underset{\sim}{F}_i + \underset{\sim}{\Upsilon} \times \underset{\sim}{F} = \underset{\sim}{M}_p + \underset{\sim}{\Upsilon} \times m\frac{d^2\underset{\sim}{r}_c}{dt^2} = \dot{\underset{\sim}{H}}_c.$$

$$\underset{\sim}{M}_p = \dot{\underset{\sim}{H}}_c + \underset{\sim}{\Upsilon} \times m\frac{d^2(\underset{\sim}{r}_p + \underset{\sim}{\Upsilon})}{dt^2} = \dot{\underset{\sim}{H}}_c + m\underset{\sim}{\Upsilon} \times (\underset{\sim}{\omega} \times \underset{\sim}{\Upsilon}) + \underset{\sim}{\Upsilon} \times m\frac{d^2\underset{\sim}{r}_p}{dt^2}$$ **, where** $\frac{d^2\underset{\sim}{\Upsilon}}{dt^2} = \underset{\sim}{\omega} \times \underset{\sim}{\Upsilon}$ **because the point moves as if it were part of the body.**

Defining $\underset{\sim}{I}_p = \underset{\sim}{I}_c + m((\underset{\sim}{\Upsilon} \bullet \underset{\sim}{\Upsilon})\underset{\sim}{\Delta} - \underset{\sim}{\Upsilon}\underset{\sim}{\Upsilon})$ **and** $\underset{\sim}{H}_p = \underset{\sim}{I}_p \bullet \underset{\sim}{\omega}$**, then** $\underset{\sim}{M}_p = \dot{\underset{\sim}{H}}_p + \underset{\sim}{\Upsilon} \times m\frac{d^2\underset{\sim}{r}_p}{dt^2}$ **. (Note: this is eqn. 8.5.8 in Baruh.)**

Since the components of $\underset{\sim}{I}_p$ **do not vary with time when expressed in the body fixed coordinate system,**

$\dot{\underset{\sim}{H}}_p = \underset{\sim}{I}_p \bullet \underset{\sim}{\alpha} + \underset{\sim}{\omega} \times \underset{\sim}{I}_p \bullet \underset{\sim}{\omega}$ **and** $\underset{\sim}{M}_p = \underset{\sim}{I}_p \bullet \underset{\sim}{\alpha} + \underset{\sim}{\omega} \times \underset{\sim}{I}_p \bullet \underset{\sim}{\omega} + \underset{\sim}{\Upsilon} \times m\frac{d^2\underset{\sim}{r}_p}{dt^2}$ **.**

General Equations for a Rigid Body

Newton's Second Law: $\underset{\sim}{F} = m\frac{d^2\underset{\sim}{r}_c}{dt^2}$.

Angular Momentum:

For the center of mass, $\dot{\underset{\sim}{H}}_c = \underset{\sim}{M}_c$**, where** $\dot{\underset{\sim}{H}}_c = \underset{\sim}{\omega} \times \underset{\sim}{I}_c \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_c \bullet \underset{\sim}{\alpha}$.

For a fixed point, $\underset{\sim}{M}_o = \underset{\sim}{\omega} \times \underset{\sim}{I}_o \bullet \underset{\sim}{\omega} + \underset{\sim}{I}_o \bullet \underset{\sim}{\alpha}$.

For an arbitrary point, $\underset{\sim}{M}_p = \underset{\sim}{I}_p \bullet \underset{\sim}{\alpha} + \underset{\sim}{\omega} \times \underset{\sim}{I}_p \bullet \underset{\sim}{\omega} + \underset{\sim}{\Upsilon} \times m\frac{d^2\underset{\sim}{r}_p}{dt^2}$.

Moment of Inertia about center of mass: $\underset{\sim}{I}_c = \underset{\sim}{e}_i(I_{im})\underset{\sim}{e}_m$ **, where** $I_{im} = \int_V \rho(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(\rho_j\rho_n)dV$.

Moment of Inertia for a body rotating about a fixed point: $\underset{\sim}{I}_o = \underset{\sim}{I}_c + m[\underset{\sim}{\Delta}(\underset{\sim}{r}_c \bullet \underset{\sim}{r}_c) - \underset{\sim}{r}_c\underset{\sim}{r}_c]$ **, where** $\underset{\sim}{r}_c$ **is the position vector from the fixed point to the center of mass and** $\underset{\sim}{\Delta}$ **is the identity tensor.**

Moment of Inertia about point moving with the body: $\underset{\sim}{I}_p = \underset{\sim}{I}_c + m((\underset{\sim}{\Upsilon} \bullet \underset{\sim}{\Upsilon})\underset{\sim}{\Delta} - \underset{\sim}{\Upsilon}\underset{\sim}{\Upsilon})$**, where** $\underset{\sim}{\Upsilon}$ **is the position vector from the point to the center of mass.**

Two Rigid Bodies in Contact

In finding equations describing the motion of a single rigid body, a point was reached where the center of mass was defined. This point depended on having a coordinate system attached to the rigid body and moving with the rigid body.

Consider the case where two rigid bodies are attached at a "point." Each rigid body can be described by its own coordinate system, attached to the body. The forces acting on each point in each body were summed. This can be done separately for each body.

$$\sum_{i=1}^{n_1}\left(\underset{\sim}{F}(y_i) + \sum_{j=1}^{n}\underset{\sim}{f}_j(y_i)\right) = \frac{d^2}{dt^2}\left(\sum_{i=1}^{n_1}\underset{\sim}{r}(dm)\right) \quad \textbf{and} \quad \sum_{i=1}^{n_2}\left(\underset{\sim}{F}(y_i) + \sum_{j=1}^{n}\underset{\sim}{f}_j(y_i)\right) = \frac{d^2}{dt^2}\left(\sum_{i=1}^{n_2}\underset{\sim}{r}(dm)\right)$$

For the internal forces, the sum runs up to $n = n_1 + n_2$, since some forces from body 2 act on body 1 and vice versa.

Splitting the sum of the internal forces and applying the definition of the center of mass for each body,

$$\left(\sum_{i=1}^{n_1}\left(\sum_{j=1}^{n_1}\underset{\sim}{f}_j(y_i) + \sum_{j=1}^{n_2}\underset{\sim}{f}_j(y_i)\right)\right) + \underset{\sim}{F}_1 = m_1\frac{d^2\underset{\sim}{r}_{c1}}{dt^2} \quad \textbf{and} \quad \left(\sum_{i=1}^{n_2}\left(\sum_{j=1}^{n_1}\underset{\sim}{f}_j(y_i) + \sum_{j=1}^{n_2}\underset{\sim}{f}_j(y_i)\right)\right) + \underset{\sim}{F}_2 = m_2\frac{d^2\underset{\sim}{r}_{c2}}{dt^2}.$$

The resultant external forces, $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ are those which are acting only on the specific body. In other words, external forces acting on body 1 do not show up in the resultant of forces acting on body 2.

Two Rigid Bodies

As before, the purely internal forces cancel and the resulting equations for each rigid body are:

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\underset{\sim}{f}_j(y_i) + \underset{\sim}{F}_1 = m_1\frac{d^2\underset{\sim}{r}_{c1}}{dt^2} \textbf{ and } \sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\underset{\sim}{f}_j(y_i) + \underset{\sim}{F}_2 = m_2\frac{d^2\underset{\sim}{r}_{c2}}{dt^2}.$$

It can readily be seen that $\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\underset{\sim}{f}_j(y_i) = -\sum_{i=1}^{n_2}\sum_{j=1}^{n_1}\underset{\sim}{f}_j(y_i) = \underset{\sim}{A}$**. This is the resultant constraint force acting at the "point" of contact. The word "point" has been put in parentheses to emphasize that many forces acting near the theoretical point of contact have been absorbed into a theoretical resultant.**

If, for instance, the point of contact is a bearing and a shaft, these internal forces would be acting over the entire inner race of the bearing. The "point" of contact would be resolved as the axis of rotation of the bearing.

The equations are now

$$\underset{\sim}{A} + \underset{\sim}{F}_1 = m_1\frac{d^2\underset{\sim}{r}_{c1}}{dt^2} \textbf{ and } -\underset{\sim}{A} + \underset{\sim}{F}_2 = m_2\frac{d^2\underset{\sim}{r}_{c2}}{dt^2}.$$

It should be noted that $\underset{\sim}{A}$ **has <u>at</u> <u>most</u> three components, and serves as a constraint to remove translational degrees of freedom from the rigid body system.**

Angular Momentum of Two Rigid Bodies

Let the origin of the coordinate system be fixed in space. Let the fixed point in each body be the center of mass. In that case, $\underset{\sim}{r}_1 = \underset{\sim}{r}_{c1} + y_i\underset{\sim}{e}_i$ **and** $\underset{\sim}{r}_2 = \underset{\sim}{r}_{c2} + y_i'\underset{\sim}{e}_i'$**.**

The angular momentum of each particle with respect to this point is: $\underset{\sim}{H}(y_i) = \underset{\sim}{r}_1 \times (dm)\frac{d\underset{\sim}{r}_1}{dt}$ **and** $\underset{\sim}{H}(y_i') = \underset{\sim}{r}_2 \times (dm)\frac{d\underset{\sim}{r}_2}{dt}$**.**

The derivative of the angular momentum is: $\frac{d\underset{\sim}{H}(y_i)}{dt} = \underset{\sim}{r}_1 \times (dm)\frac{d^2\underset{\sim}{r}_1}{dt^2}$ **and** $\frac{d\underset{\sim}{H}(y_i')}{dt} = \underset{\sim}{r}_2 \times (dm)\frac{d^2\underset{\sim}{r}_2}{dt^2}$**.**

Using the EOM for the mass, $\underset{\sim}{F}(y_i) + \sum_{j=1}^{n}\underset{\sim}{f}_j(y_i) = (dm)\frac{d^2(\underset{\sim}{r}_1)}{dt^2}$ **and** $\underset{\sim}{F}(y_i') + \sum_{j=1}^{n}\underset{\sim}{f}_j(y_i') = (dm)\frac{d^2(\underset{\sim}{r}_2)}{dt^2}$**,**

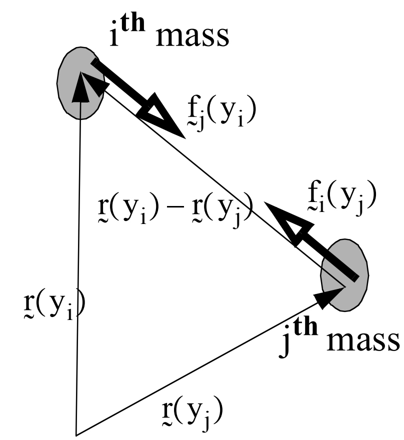
$$\frac{d\underset{\sim}{H}(y_i)}{dt} = \underset{\sim}{r}_1 \times \left[\underset{\sim}{F}(y_i) + \sum_{j=1}^{n}\underset{\sim}{f}_j(y_i)\right] \quad \textbf{and} \quad \frac{d\underset{\sim}{H}(y_i')}{dt} = \underset{\sim}{r}_2 \times \left[\underset{\sim}{F}(y_i') + \sum_{j=1}^{n}\underset{\sim}{f}_j(y_i')\right].$$

Summing up the angular momenta yields the angular momentum derivative for the entire body,

$$\frac{d(\underset{\sim}{H}_{o1})}{dt} = \underset{\sim}{M}_{o1} + \left(\sum_{i=1}^{n_1}\underset{\sim}{r}_1 \times \left[\sum_{j=1}^{n_1}\underset{\sim}{f}_j(y_i)\right]\right) + \left(\sum_{i=1}^{n_1}\underset{\sim}{r}_1 \times \left[\sum_{j=1}^{n_2}\underset{\sim}{f}_j(y_i)\right]\right) \quad \textbf{and} \quad \frac{d(\underset{\sim}{H}_{o2})}{dt} = \underset{\sim}{M}_{o2} + \left(\sum_{i=1}^{n_2}\underset{\sim}{r}_2 \times \left[\sum_{j=1}^{n_2}\underset{\sim}{f}_j(y_i')\right]\right) + \left(\sum_{i=1}^{n_2}\underset{\sim}{r}_2 \times \left[\sum_{j=1}^{n_1}\underset{\sim}{f}_j(y_i')\right]\right).$$

Moment Constraints at the Point of Contact

As before, the forces internal to each rigid body have balanced moments and



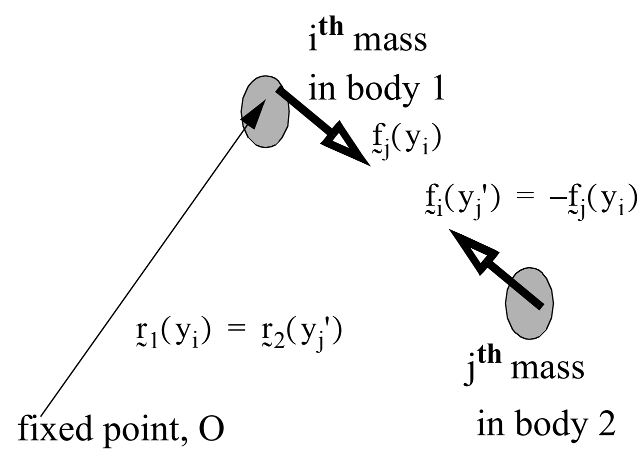
$$\sum_{i=1}^{n_1} \underset{\sim}{r}_1 \times \left[\sum_{j=1}^{n_1} \underset{\sim}{f}_j(y_i)\right] = 0 \textbf{ and } \sum_{i=1}^{n_2} \underset{\sim}{r}_2 \times \left[\sum_{j=1}^{n_2} \underset{\sim}{f}_j(y_i')\right] = 0 .$$

In the first body, the forces exerted by the second body are zero except at the common "point" of contact. At this common "point", the position vectors are the same.

In other words, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$ **at the common "point" of contact. Because of the equal and opposite assumption,**

$$\left(\sum_{i=1}^{n_1} \underset{\sim}{r}_1 \times \left[\sum_{j=1}^{n_2} \underset{\sim}{f}_j(y_i)\right]\right) = -\left(\sum_{i=1}^{n_2} \underset{\sim}{r}_2 \times \left[\sum_{j=1}^{n_1} \underset{\sim}{f}_j(y_i')\right]\right) = \underset{\sim}{M}_A .$$

The EOM are $\frac{d(\underset{\sim}{H}_{o1})}{dt} = \underset{\sim}{M}_{o1} + \underset{\sim}{M}_A$ **and** $\frac{d(\underset{\sim}{H}_{o2})}{dt} = \underset{\sim}{M}_{o2} - \underset{\sim}{M}_A$



Summary of Two Rigid Bodies

With two rigid bodies, there are six degrees of freedom. These six degrees of freedom are described by the six equations of motion:

$$\underset{\sim}{A} + \underset{\sim}{F}_1 = m_1 \frac{d^2 \underset{\sim}{r}_{c1}}{dt^2} \text{ and } -\underset{\sim}{A} + \underset{\sim}{F}_2 = m_2 \frac{d^2 \underset{\sim}{r}_{c2}}{dt^2}$$

$$\frac{d(\underset{\sim}{H}_{o1})}{dt} = \underset{\sim}{M}_{o1} + \underset{\sim}{M}_A \text{ and } \frac{d(\underset{\sim}{H}_{o2})}{dt} = \underset{\sim}{M}_{o2} - \underset{\sim}{M}_A$$

If the constraint force and the contraint moment have all components, then the two rigid bodies are rigidly constrained and the situation has returned to a single rigid body.

Based on the type of constraint, there will be missing terms in either the force or the moment. This will determine how many degrees of freedom the system contains.

Two Rigid Bodies, Multiple Constraints

If there are multiple “points” of contact, these equations can be extended without much effort to:

$$\sum_{i=1}^{n_p} \underset{\sim}{A}_i + \underset{\sim}{F}_1 = m_1 \frac{d^2 \underset{\sim}{r}_{c1}}{dt^2} \quad \textbf{and} \quad -\sum_{i=1}^{n_p} \underset{\sim}{A}_i + \underset{\sim}{F}_2 = m_2 \frac{d^2 \underset{\sim}{r}_{c2}}{dt^2}$$

$$\frac{d(\underset{\sim}{H}_{o1})}{dt} = \underset{\sim}{M}_{o1} + \sum_{i=1}^{n_p} \underset{\sim}{M}_{A;i} \quad \textbf{and} \quad \frac{d(\underset{\sim}{H}_{o2})}{dt} = \underset{\sim}{M}_{o2} - \sum_{i=1}^{n_p} \underset{\sim}{M}_{A;i}$$

There is a down-side to this set of equations, the two rigid bodies may now be subject to more constraints than there are degrees of freedom. Only six independent constraints can exist. In fact, only three translational constraints and three rotational constraints can exist.

If there are more constraints than there are degrees of freedom, the system is over-constrained.

It is beyond the scope of this section to indicate how to deal with this problem. However, in general, there will be relations relating the non-independent constraints.

Multiple Rigid Bodies

Extending the method for two rigid bodies to multiple rigid bodies is straight-forward.

Consider N rigid bodies. There are 2N vector equations of the form:

$$\sum_{i=1}^{n_{p;k}} \underset{\sim}{A}_{ik} + \underset{\sim}{F}_k = m_k \frac{d^2 \underset{\sim}{r}_{c;k}}{dt^2}$$

$$\frac{d(\underset{\sim}{H}_{o;k})}{dt} = \underset{\sim}{M}_{o;k} + \sum_{i=1}^{n_{p;k}} \underset{\sim}{M}_{A;ik}$$

The constraints applied to a single rigid body may be applied by multiple rigid bodies.

$$\underset{\sim}{a}_{(\zeta+1)} = \underset{\sim}{a}_{(\zeta)} + \underset{\sim}{\alpha}_{(\zeta+1)} \times \underset{\sim}{r}_{(\zeta+1)} + \underset{\sim}{a}_{rot(\zeta+1)} + 2\underset{\sim}{\omega}_{(\zeta+1)} \times \underset{\sim}{v}_{rot(\zeta+1)} + \underset{\sim}{\omega}_{(\zeta+1)} \times [\underset{\sim}{\omega}_{(\zeta+1)} \times \underset{\sim}{r}_{(\zeta+1)}]$$

$$\underset{\sim}{a}_{(1)} = \underset{\sim}{a}_{(0)} + \underset{\sim}{\alpha}_{(1)} \times \underset{\sim}{r}_{(1)} + \underset{\sim}{a}_{rot(1)} + 2\underset{\sim}{\omega}_{(1)} \times \underset{\sim}{v}_{rot(1)} + \underset{\sim}{\omega}_{(1)} \times [\underset{\sim}{\omega}_{(1)} \times \underset{\sim}{r}_{(1)}]$$

Treating Non-flat Ground

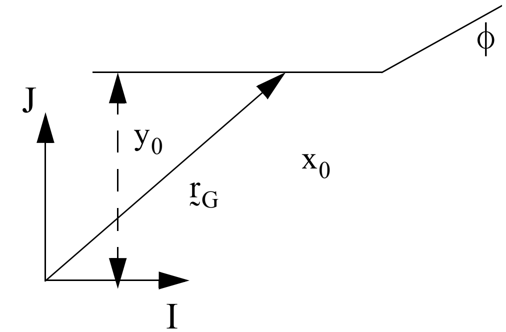
In the case of a one dimensional problem, where the ground is not flat, a body fixed coordinate system must be developed. This coordinate system maintains the ground's tangent and normal directions.

The location of the ground can be described by a vector function of the wheel's x coordinate, $\underset{\sim}{r}_G = \chi(x)\underset{\sim}{I} + \psi(x)\underset{\sim}{J}$**.**

The vector which is tangent to this ground curve is $\underset{\sim}{T} = \left(\frac{d\chi(x)}{dx}\right)\underset{\sim}{I} + \left(\frac{d\psi(x)}{dx}\right)\underset{\sim}{J}$**. The unit tangent vector is** $\underset{\sim}{i} = \frac{\underset{\sim}{T}}{\|\underset{\sim}{T}\|}$**.**

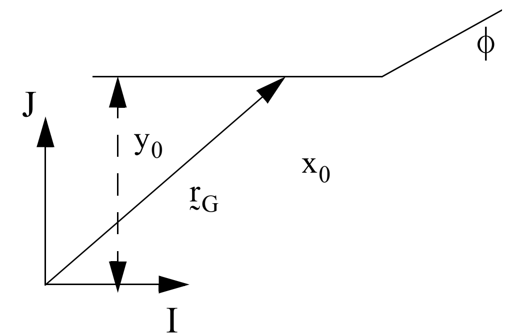
The outward normal vector to this ground curve is $\underset{\sim}{N} = \underset{\sim}{k} \times \underset{\sim}{T} = K \times \left[\left(\frac{d\chi(x)}{dx}\right)\underset{\sim}{I} + \left(\frac{d\psi(x)}{dx}\right)\underset{\sim}{J}\right] = -\left(\frac{d\psi(x)}{dx}\right)\underset{\sim}{I} + \left(\frac{d\chi(x)}{dx}\right)\underset{\sim}{J}$**.**

The unit vector is $\underset{\sim}{j} = \dfrac{\underset{\sim}{N}}{\|\underset{\sim}{N}\|}$.



Example: Simple Slope

For instance, a simple slope would have the function



$$\underset{\sim}{r}_G = \begin{cases} x\underset{\sim}{I} + y_0\underset{\sim}{J} & x < x_0 \\ x\underset{\sim}{I} + (y_0 + (x - x_0)\tan\phi)\underset{\sim}{J} & x \ge x_0 \end{cases}.$$

The tangent vector is $\underset{\sim}{T} = \begin{cases} \underset{\sim}{I} & x < x_0 \\ \underset{\sim}{I} + \tan\phi\underset{\sim}{J} & x \ge x_0 \end{cases}$ **and the tangent unit vector is**

$$\underset{\sim}{i} = \begin{cases} \underset{\sim}{I} & x < x_0 \\ \cos\phi\underset{\sim}{I} + \sin\phi\underset{\sim}{J} & x \ge x_0 \end{cases}.$$

The normal vector is $\underset{\sim}{N} = \begin{cases} \underset{\sim}{J} & x < x_0 \\ \underset{\sim}{J} - \tan\phi\underset{\sim}{I} & x \ge x_0 \end{cases}$ **and the normal unit vector is** $\underset{\sim}{j} = \begin{cases} \underset{\sim}{J} & x < x_0 \\ \cos\phi\underset{\sim}{J} - \sin\phi\underset{\sim}{I} & x \ge x_0 \end{cases}$.

Ignoring the instantaneous change at the change in slope and the multiple contact points when a wheel initially intersects the slope, the angular velocity for this coordinate transformation matrix is zero.

Example: Wheel Rolling up a Slope

The forces acting on the wheel are:
$$\underset{\sim}{F}_G = N\underset{\sim}{j} + F_f\underset{\sim}{i}$$
$$\underset{\sim}{W} = -mg\underset{\sim}{J}$$
$$\underset{\sim}{\tau} = \tau\underset{\sim}{K}$$

A vector pointing from the wheel COM to the point of contact is: $\underset{\sim}{r} = -r\underset{\sim}{j}$**.**

The vector pointing from the starting point to the center of the wheel is $\underset{\sim}{r}_C = \chi(x)\underset{\sim}{I} + \psi(x)\underset{\sim}{J}$**, where**

$$\chi(x) = \begin{cases} x & x < x_0 \\ x_0 + (x - x_0)\cos\phi & x \ge x_0 \end{cases} \quad \textbf{and} \quad \psi(x) = \begin{cases} 0 & x < x_0 \\ (x - x_0)\sin\phi & x \ge x_0 \end{cases}.$$

The acceleration of the center of mass of the wheel is $\underset{\sim}{a} = \begin{cases} \ddot{x}\underset{\sim}{I} & x < x_0 \\ \ddot{x}(\cos\phi\underset{\sim}{I} + \sin\phi\underset{\sim}{J}) & x \ge x_0 \end{cases}$**.**

By definition of the unit vector, $\underset{\sim}{a} = \ddot{x}\underset{\sim}{i}$**.**

Example: Wheel Rolling up a Slope

The angular velocity of a coordinate system rotating with the wheel is $\underset{\sim}{\omega} = \dot{\theta}\underset{\sim}{k}'$**.**

The moment of inertia tensor in body fixed coordinates is: $\underset{\sim}{I} = I_w(\underset{\sim}{i}'\underset{\sim}{i}' + \underset{\sim}{j}'\underset{\sim}{j}') + J_w\underset{\sim}{k}'\underset{\sim}{k}'$.

The angular momentum about the center of mass is: $\underset{\sim}{H}_{cm} = \underset{\sim}{I} \bullet \underset{\sim}{\omega} = [I_w(\underset{\sim}{i}'\underset{\sim}{i}' + \underset{\sim}{j}'\underset{\sim}{j}') + J_w\underset{\sim}{k}'\underset{\sim}{k}'] \bullet \dot{\theta}\underset{\sim}{k}' = J_w\dot{\theta}\underset{\sim}{k}' = J_w\dot{\theta}\underset{\sim}{K}$. **The derivative of this yields:** $\dot{\underset{\sim}{H}}_{cm} = J_w\ddot{\theta}\underset{\sim}{K}$.

Summing forces yields: $\underset{\sim}{F}_G + \underset{\sim}{W} = N\underset{\sim}{j} + F_f\underset{\sim}{i} - mg\underset{\sim}{J} = m\underset{\sim}{a} = m\ddot{x}\underset{\sim}{i}$.

The vector $\underset{\sim}{J} = \begin{cases} \underset{\sim}{j} & x < x_0 \\ \sin\phi\underset{\sim}{i} + \cos\phi\underset{\sim}{j} & x \geq x_0 \end{cases}$.

This yields two solutions:

$$\begin{array}{lll} m\ddot{x} = F_f & N = mg & x < x_0 \\ m\ddot{x} = F_f - mg\sin\phi & N = mg\cos\phi & x \geq x_0 \end{array}.$$