Coordinate Transformation Matrix and Angular Velocity … vector formulation

Consider a vector rotating in time, $\vec{r}(t)=r_0(\cos(\theta t)\hat{i}+\sin(\theta t)\hat{j})$.

Define a unit vector, $\hat{e}_1(t)=\cos(\theta t)\hat{i}+\sin(\theta t)\hat{j}$.

This can be checked for magnitude by $\|e_1(t)\|=\sqrt{\cos^2(\theta t)+\sin^2(\theta t)}=1$.

In $\hat{i}$, $\hat{j}$ coordinates, all the time-changing occurs on the components, $r_x(t)$, $r_y(t)$. In the $\hat{e}_1(t)$ coordinate, all the time-changing occurs with the base vector.

Taking the time derivative to get the velocity: $\vec{v}(t)=r_0\dfrac{d\hat{e}_1(t)}{dt}$.

The derivative of the base vector yields: $\dfrac{d\hat{e}_1(t)}{dt}=\dot{\theta}(-\sin(\theta t)\hat{i}+\cos(\theta t)\hat{j})$.

The new base vector, $\hat{e}_2(t)=-\sin(\theta t)\hat{i}+\cos(\theta t)\hat{j}$ is perpendicular to the original base vector, $\hat{e}_1(t)$, which can be shown by dotting the two vectors,

$$\hat{e}_1(t)\cdot\hat{e}_2(t)=-\cos(\theta t)\sin(\theta t)+\sin(\theta t)\cos(\theta t)=0$$

Two orthonormal vectors can form a basis for the two dimensional space just as well as the $\hat{i}$, $\hat{j}$ pair.

It is possible to transform between the two bases. In matrix notation,

$$\begin{bmatrix}\hat{e}_1(t)\\ \hat{e}_2(t)\end{bmatrix}=\begin{bmatrix}\cos(\theta t) & \sin(\theta t)\\ -\sin(\theta t) & \cos(\theta t)\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\end{bmatrix}$$

Call $[A]=\begin{bmatrix}\cos(\theta t) & \sin(\theta t)\\ -\sin(\theta t) & \cos(\theta t)\end{bmatrix}$ a coordinate transformation matrix (CTM).

What does $[A]^T[A]$ look like? $[A]^T[A]=\begin{bmatrix}\cos(\theta t) & -\sin(\theta t)\\ \sin(\theta t) & \cos(\theta t)\end{bmatrix}\begin{bmatrix}\cos(\theta t) & \sin(\theta t)\\ -\sin(\theta t) & \cos(\theta t)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

What does $[A][A]^T$ look like? $[A][A]^T=\begin{bmatrix}\cos(\theta t) & \sin(\theta t)\\ -\sin(\theta t) & \cos(\theta t)\end{bmatrix}\begin{bmatrix}\cos(\theta t) & -\sin(\theta t)\\ \sin(\theta t) & \cos(\theta t)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

Hence, $[A]^T=[A]^{-1}$.

This is generally true for a coordinate transformation matrix and will be shown later.

What does $det([A])$ look like? $det([A])=\cos^2(\theta t)+\sin^2(\theta t)=1$

Transforming from right-handed CS to right-handed CS or left-handed CS to left-handed CS will result in a CTM with determinant equal to one. Otherwise, the determinant will be negative one.

What are the components of the Coordinate Transformation Matrix?

The dot product between two vectors is $\vec{A}\cdot\vec{B}=(\|\vec{A}\|)(\|\vec{B}\|)\cos\theta$ where θ is the angle between the two vectors. For unit vectors, $\hat{e}_1\cdot\hat{e}_2=\cos\theta$ because the magnitudes are 1.

For a general Coordinate Transformation Matrix, $[A]=\begin{bmatrix}\hat{e}_1\cdot\hat{i} & \hat{e}_1\cdot\hat{j} & \hat{e}_1\cdot\hat{k}\\ \hat{e}_2\cdot\hat{i} & \hat{e}_2\cdot\hat{j} & \hat{e}_2\cdot\hat{k}\\ \hat{e}_3\cdot\hat{i} & \hat{e}_3\cdot\hat{j} & \hat{e}_3\cdot\hat{k}\end{bmatrix}$, it can be seen that the entries in the matrix are the cosines of the angles between the vectors themselves.

Applying the transformation to the original base vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$, gives,

$$\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ e_3\end{bmatrix}=\begin{bmatrix}\hat{e}_1\cdot\hat{i} & \hat{e}_1\cdot\hat{j} & \hat{e}_1\cdot\hat{k}\\ \hat{e}_2\cdot\hat{i} & \hat{e}_2\cdot\hat{j} & \hat{e}_2\cdot\hat{k}\\ \hat{e}_3\cdot\hat{i} & \hat{e}_3\cdot\hat{j} & \hat{e}_3\cdot\hat{k}\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix} .$$

This can be verified by considering the first row,

$\hat{e}_1=\begin{bmatrix}\hat{e}_1\cdot\hat{i} & \hat{e}_1\cdot\hat{j} & \hat{e}_1\cdot\hat{k}\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}=(\hat{e}_1\cdot\hat{i})\hat{i}+(\hat{e}_1\cdot\hat{j})\hat{j}+(\hat{e}_1\cdot\hat{k})\hat{k}$. Dot this equality with $\hat{i}$, $\hat{j}$, $\hat{k}$ to see that the relationship is correct.

Consider $[A][A]^T$ more closely.

The first entry is $a_{11}=\begin{bmatrix}\hat{e}_1\cdot\hat{i} & \hat{e}_1\cdot\hat{j} & \hat{e}_1\cdot\hat{k}\end{bmatrix}\begin{bmatrix}\hat{e}_1\cdot\hat{i}\\ \hat{e}_1\cdot\hat{j}\\ \hat{e}_1\cdot\hat{k}\end{bmatrix}=(\hat{e}_1\cdot\hat{i})^2+(\hat{e}_1\cdot\hat{j})^2+(\hat{e}_1\cdot\hat{k})^2$. That's not very helpful.

Try dotting $\hat{e}_1=(\hat{e}_1\cdot\hat{i})\hat{i}+(\hat{e}_1\cdot\hat{j})\hat{j}+(\hat{e}_1\cdot\hat{k})\hat{k}$ with $\hat{e}_1$,

$\hat{e}_1\cdot\hat{e}_1=(\hat{e}_1\cdot\hat{i})(\hat{i}\cdot\hat{e}_1)+(\hat{e}_1\cdot\hat{j})(\hat{j}\cdot\hat{e}_1)+(\hat{e}_1\cdot\hat{k})(\hat{k}\cdot\hat{e}_1)$. Since $\hat{e}_1\cdot\hat{e}_1=1$ and the dot product is commutative, it can be seen that the first entry in $[A][A]^T$ is 1.

This procedure can be followed with the remaining entries in the CTM to discover that $[A][A]^T=[I]$.

$$a_{12}=\begin{bmatrix}\hat{e}_1\cdot\hat{i} & \hat{e}_1\cdot\hat{j} & \hat{e}_1\cdot\hat{k}\end{bmatrix}\begin{bmatrix}\hat{e}_2\cdot\hat{i}\\ \hat{e}_2\cdot\hat{j}\\ \hat{e}_2\cdot\hat{k}\end{bmatrix}=(\hat{e}_1\cdot\hat{i})(\hat{e}_2\cdot\hat{i})+(\hat{e}_1\cdot\hat{j})(\hat{e}_2\cdot\hat{j})+(\hat{e}_1\cdot\hat{k})(\hat{e}_2\cdot\hat{k})=\hat{e}_1\cdot\hat{e}_2=0 .$$

$$a_{13}=\begin{bmatrix}\hat{e}_1\cdot\hat{i} & \hat{e}_1\cdot\hat{j} & \hat{e}_1\cdot\hat{k}\end{bmatrix}\begin{bmatrix}\hat{e}_3\cdot\hat{i}\\ \hat{e}_3\cdot\hat{j}\\ \hat{e}_3\cdot\hat{k}\end{bmatrix}=(\hat{e}_1\cdot\hat{i})(\hat{e}_3\cdot\hat{i})+(\hat{e}_1\cdot\hat{j})(\hat{e}_3\cdot\hat{j})+(\hat{e}_1\cdot\hat{k})(\hat{e}_3\cdot\hat{k})=\hat{e}_1\cdot\hat{e}_3=0 .$$

And so on.

The opposite relationship can be shown by noting that $\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \hat{i}\cdot\hat{e}_1 & \hat{j}\cdot\hat{e}_1 & \hat{k}\cdot\hat{e}_1 \\ \hat{i}\cdot\hat{e}_2 & \hat{j}\cdot\hat{e}_2 & \hat{k}\cdot\hat{e}_2 \\ \hat{i}\cdot\hat{e}_3 & \hat{j}\cdot\hat{e}_3 & \hat{k}\cdot\hat{e}_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix}$, which can be verified by dotting each row with $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$.

Since $\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = [A]^T \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix}$ and $[A][A]^T = [I]$, $[A]^T[A] = [I]$.

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = [A]^T \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix} = [A]^T [I] \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix} = [A]^T ([A][A]^T) \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix} = ([A]^T[A])[A]^T \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix} = [I][A]^T \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ e_3 \end{bmatrix} .$$

Cute trick with CTM

Since the CTM is a special case of a unitary matrix (the general unitary matrix has complex elements), $[A][A]^T=[I]$.

Taking the time derivative of this relation gives $\frac{d}{dt}([A][A]^T=[I])$

Hence, $\frac{d}{dt}([A])[A]^T+[A]\frac{d}{dt}([A]^T)=[0]$.

Or, $\frac{d}{dt}([A])[A]^T=-[A]\frac{d}{dt}([A]^T)=-(\frac{d}{dt}([A])[A]^T)^T$.

Defining $[G]=\frac{d}{dt}([A])[A]^T$, it can be seen that $[G]=-[G]^T$ which is the definition of a skew-symmetric matrix.

Blowing this up, $\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}=\begin{bmatrix} -g_{11} & -g_{21} & -g_{31} \\ -g_{12} & -g_{22} & -g_{32} \\ -g_{13} & -g_{23} & -g_{33} \end{bmatrix}$, which gives one important piece of information, that $\begin{matrix} g_{11}=-g_{11}=0 \\ g_{22}=-g_{22}=0 \\ g_{33}=-g_{33}=0 \end{matrix}$. Using the cyclic permutation, $\begin{matrix} g_{23}=\Omega_1 \\ g_{31}=\Omega_2 \\ g_{12}=\Omega_3 \end{matrix}$ to define $\Omega_{1,}\Omega_{2,}\Omega_3$ gives the matrix $[G]=\begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix}$.

Defining the angular velocity vector by assembling the components, $\Omega_{1,}\Omega_{2,}\Omega_3$, with the transformed base vectors, $\vec{\Omega}=[\Omega_1 \quad \Omega_2 \quad \Omega_3]\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$.

Return to the problem where $\vec{r}=r'_x\hat{e}_1+r'_y\hat{e}_2+r'_z\hat{e}_3$ and where r'_x, r'_y, r'_z are invariant with time.

The components in the original coordinate system are $\begin{bmatrix} r'_x \\ r'_y \\ r'_z \end{bmatrix}=[A]\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$.

Then, $\vec{v}=\frac{d}{dt}(r'_x\hat{e}_1+r'_y\hat{e}_2+r'_z\hat{e}_3)=r'_x\frac{d\hat{e}_1}{dt}+r'_y\frac{d\hat{e}_2}{dt}+r'_z\frac{d\hat{e}_3}{dt}$ by the time invariance of r'_x, r'_y, r'_z .

Writing this in matrix notation, $\vec{v}=\begin{bmatrix} r'_x & r'_y & r'_z \end{bmatrix}\frac{d}{dt}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$.

The CTM can be used to evaluate $\frac{d}{dt}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}=\frac{d[A]}{dt}\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}=\frac{d[A]}{dt}[A]^T\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}=[G]\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$.

Hence, $\vec{v}=\begin{bmatrix} r'_x & r'_y & r'_z \end{bmatrix}[G]\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}=\begin{bmatrix} r'_x & r'_y & r'_z \end{bmatrix}\begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$

Continuing to reduce,

$$\vec{v}=\begin{bmatrix} r'_x(\Omega_3\hat{e}_2-\Omega_2\hat{e}_3) \\ r'_y(\Omega_1\hat{e}_3-\Omega_3\hat{e}_1) \\ r'_z(\Omega_2\hat{e}_1-\Omega_1\hat{e}_2) \end{bmatrix}=\begin{bmatrix} r'_y\Omega_3-r'_z\Omega_2 & r'_x\Omega_3-r'_z\Omega_1 & r'_y\Omega_1-r'_x\Omega_2 \end{bmatrix}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}=\vec{\Omega}\times\vec{r}$$

The point behind this exercise, is that evaluating time derivatives has been converted to evaluating cross products, a much less error-prone operation.

The angular velocity, a concept that will become much more powerful in dynamics of rigid bodies, has also been introduced here, along with a simple method of calculating it based on the coordinate transformation matrix.

The first steps in setting up a dynamics problem are to write down all the coordinate systems involved, to write down all the vectors to be involved, and to calculate the angular velocity for each coordinate system.

It will turn out later that chaining coordinate transformations leads to a simple formulation.

The angular acceleration,

$$\vec{\alpha}=\frac{d\vec{\Omega}}{dt}=\begin{bmatrix}\dot{\Omega}_1 & \dot{\Omega}_2 & \dot{\Omega}_3\end{bmatrix}\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}+\begin{bmatrix}\Omega_1 & \Omega_2 & \Omega_3\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}=\begin{bmatrix}\dot{\Omega}_1 & \dot{\Omega}_2 & \dot{\Omega}_3\end{bmatrix}\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}+\vec{\Omega}\times\vec{\Omega}=\begin{bmatrix}\dot{\Omega}_1 & \dot{\Omega}_2 & \dot{\Omega}_3\end{bmatrix}\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix} \quad .$$

Back to the original problem that spawned this aside …

$$\vec{r}(t)=r_0(\cos(\theta t)\hat{i}+\sin(\theta t)\hat{j})=r_0\hat{e}_1 \quad .$$

The coordinate transformation matrix is $[A]=\begin{bmatrix}\cos(\theta t) & \sin(\theta t) & 0\\ -\sin(\theta t) & \cos(\theta t) & 0\\ 0 & 0 & 1\end{bmatrix}$.

From the CTM,

$$[G]=\frac{d}{dt}[A][A]^T=\dot{\theta}\begin{bmatrix}-\sin(\theta t) & \cos(\theta t) & 0\\ -\cos(\theta t) & -\sin(\theta t) & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos(\theta t) & -\sin(\theta t) & 0\\ \sin(\theta t) & \cos(\theta t) & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & \dot{\theta} & 0\\ -\dot{\theta} & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad .$$

Therefore, $\vec{\Omega}=\dot{\theta}\hat{e}_3$ and $\vec{v}=\vec{\Omega}\times\vec{r}=\dot{\theta}\hat{e}_3\times r_0\hat{e}_1=r_0\dot{\theta}\hat{e}_2=r_0\dot{\theta}(-\sin(\theta t)\hat{i}+\cos(\theta t)\hat{j})$.

Consider $\vec{r}(t)=\begin{bmatrix} r_x(t) & r_y(t) & r_z(t) \end{bmatrix}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}$.

The velocity is $\vec{v}(t)=\frac{d\,\vec{r}(t)}{dt}=\begin{bmatrix} \dot{r}_x(t) & \dot{r}_y(t) & \dot{r}_z(t) \end{bmatrix}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}+\begin{bmatrix} r_x(t) & r_y(t) & r_z(t) \end{bmatrix}\frac{d}{dt}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}$.

Define the "velocity as seen from the rotating coordinate system" as

$$\vec{v_{rot}}(t)=\frac{d\,\vec{r}(t)}{dt}=\begin{bmatrix} \dot{r}_x(t) & \dot{r}_y(t) & \dot{r}_z(t) \end{bmatrix}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}$$

and substitute the earlier result, $\begin{bmatrix} r'_x & r'_y & r'_z \end{bmatrix}\frac{d}{dt}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}=\vec{\Omega}\times\vec{r}$, and the velocity is

$$\vec{v}(t)=\frac{d\,\vec{r}(t)}{dt}=\vec{v_{rot}}(t)+\vec{\Omega}\times\vec{r} \quad .$$

The acceleration comes from this equation:

$$\vec{a}(t)=\frac{d\,\vec{v}(t)}{dt}=\frac{d}{dt}(\vec{v_{rot}}(t)+\vec{\Omega}\times\vec{r})=\frac{d\,\vec{v_{rot}}}{dt}+\frac{d\vec{\Omega}}{dt}\times\vec{r}+\vec{\Omega}\times\frac{d\,\vec{r}}{dt}$$

The first term is $\frac{d\,\vec{v_{rot}}}{dt}=\begin{bmatrix} \ddot{r}_x(t) & \ddot{r}_y(t) & \ddot{r}_z(t) \end{bmatrix}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}+\begin{bmatrix} \dot{r}_x(t) & \dot{r}_y(t) & \dot{r}_z(t) \end{bmatrix}\frac{d}{dt}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}$. Defining the "acceleration as seen in the rotating coordinate system" as $\vec{a_{rot}}=\begin{bmatrix} \ddot{r}_x(t) & \ddot{r}_y(t) & \ddot{r}_z(t) \end{bmatrix}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}$ and noting that $\begin{bmatrix} \dot{r}_x(t) & \dot{r}_y(t) & \dot{r}_z(t) \end{bmatrix}\frac{d}{dt}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}=\vec{\Omega}\times\vec{v_{rot}}$, $\frac{d\,\vec{v_{rot}}}{dt}=\vec{a_{rot}}+\vec{\Omega}\times\vec{v_{rot}}$.

The second term can be reduced by noting that

$$\frac{d\vec{\Omega}}{dt}=\begin{bmatrix} \frac{d\,\Omega_1}{dt} & \frac{d\,\Omega_2}{dt} & \frac{d\,\Omega_3}{dt} \end{bmatrix}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}+\begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 \end{bmatrix}\frac{d}{dt}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}$$

Defining $\vec{\alpha}=\begin{bmatrix} \frac{d\,\Omega_1}{dt} & \frac{d\,\Omega_2}{dt} & \frac{d\,\Omega_3}{dt} \end{bmatrix}\begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{e}_3(t) \end{bmatrix}$, the angular acceleration and noting that

$$\begin{bmatrix}\Omega_1 & \Omega_2 & \Omega_3\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\hat{e}_1(t)\\ \hat{e}_2(t)\\ \hat{e}_3(t)\end{bmatrix}=\vec{\Omega}\times\Omega=\hat{0} \quad , \quad \frac{d\vec{\Omega}}{dt}\times\vec{r}=\vec{\alpha}\times\vec{r}$$

The third term reduces

$$\vec{\Omega}\times\frac{d\vec{r}}{dt}=\vec{\Omega}\times(\vec{v_{rot}}+\vec{\Omega}\times\vec{r})=\vec{\Omega}\times\vec{v_{rot}}+\Omega\times(\Omega\times\vec{r}) \quad .$$

Putting it all together gives the formula for an acceleration expressed in a rotating coordinate system,

$$\vec{a}(t)=\vec{a_{rot}}+\vec{\Omega}\times\vec{v_{rot}}+\vec{\alpha}\times\vec{r}+\vec{\Omega}\times\vec{v_{rot}}+\Omega\times(\Omega\times\vec{r})$$

Combining terms gives

$$\vec{a}(t)=\vec{a_{rot}}+2\vec{\Omega}\times\vec{v_{rot}}+\vec{\alpha}\times\vec{r}+\Omega\times(\Omega\times\vec{r})$$

Consider a vector, $\vec{r}=\vec{r}_0+\vec{r}_1$.

The velocity is $\vec{v}=\vec{v}_0+\vec{v}_1$ and the acceleration is $\vec{a}=\vec{a}_0+\vec{a}_1$.

If $\vec{r}_1$ is expressed in a rotating coordinate system, then the general formulas for velocity and acceleration are:

$$\vec{v}(t)=\vec{v}_0+(\vec{v}_1)_{rot}+\vec{\Omega}\times\vec{r}_1$$

and

$$\vec{a}(t)=\vec{a}_0+(\vec{a}_1)_{rot}+2\vec{\Omega}\times(\vec{v}_1)_{rot}+\vec{\alpha}\times\vec{r}_1+\Omega\times(\Omega\times\vec{r}_1)$$