From Dr. Ann Wright's (Hendrix College) Vectors & Matrix Basics

I. Vectors

A vector is represented in terms of its basis vectors, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and its components, (a_1, a_2, a_3), by the linear combination, $\vec{A} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$. Using rules of matrix multiplication, this can be written as a product of a row vector and a column vector, so $\vec{A} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$. Using matrix notation can make many of the computations much easier to process. Because of commutativity of vector operations, the vector can also be written as $\vec{A} = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

The dot product between two vectors can be processed using vector notation,

$$\vec{A} \cdot \vec{B} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$, where $$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ leads to

$$\vec{A} \cdot \vec{B} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

II. Matrix Basics

Let $M=\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$, $N=\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$, and $T=\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$

1. Determinant of a 2x2 matrix

$$det\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}=t_{11}t_{22}-t_{21}t_{12}$$

2. Cofactor matrix

$$(M_{cof})_{ij}=(-1)^{i+j}\times det\left[\text{matrix left by crossing out i}^{\text{th}}\text{ row and j}^{\text{th}}\text{column}\right]$$

example: $(M_{cof})_{11}=m_{22}m_{33}-m_{23}m_{32}$

3. Determinant of a 3x3 matrix

$$det(M)=m_{11}(M_{cof})_{11}+m_{12}(M_{cof})_{12}+m_{13}(M_{cof})_{13} =$$
$$m_{11}(m_{22}m_{33}-m_{23}m_{32})-m_{12}(m_{21}m_{33}-m_{23}m_{31})+m_{13}(m_{21}m_{32}-m_{22}m_{31})$$

4. Matrix addition

$$M+N=\begin{bmatrix} m_{11}+n_{11} & m_{12}+n_{12} & m_{13}+n_{13} \\ m_{21}+n_{21} & m_{22}+n_{22} & m_{23}+n_{23} \\ m_{31}+n_{31} & m_{32}+n_{32} & m_{33}+n_{33} \end{bmatrix}$$

5. Multiplication of a matrix and a vector

$$\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix} M \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}\cdot\vec{A}=\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}=\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix}\begin{bmatrix} m_{11}a_1+m_{12}a_2+m_{13}a_3 \\ m_{21}a_1+m_{22}a_2+m_{23}a_3 \\ m_{31}a_1+m_{32}a_2+m_{33}a_3 \end{bmatrix}$$

6. Multiplication of 2 matrices

$$MN=\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

<more to come ...>

7. Multiplication of a matrix and a basis vector

<stay tuned for future draft>

8. Identity matrix

$$I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Matrix Transpose

$$M^T=\begin{bmatrix} m_{11} & m_{21} & m_{31} \\ m_{12} & m_{22} & m_{32} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}$$

10. Matrix Inverse

$$M\,M^{-1} = M^{-1}\,M = I$$

<there is a co-factor formula, but that will not be presented at this time>

The inverse of a 2x2 matrix is:

$$T^{-1} = \frac{1}{det(T)} \begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix}$$

11. Matrix Conjugate

<not relevant in this course. Implemented later>

12. Matrix Adjoint

<not relevant in this course. Implemented later>

13. Commutation

<not relevant in this course. Implemented later>

14. Product Rules

$$(MN)^T = N^T\,M^T$$

$$(MN)^{-1} = N^{-1}\,M^{-1}$$

15. Cross Product Calculation

The cross product $\vec{C} = \vec{A} \times \vec{B}$ can be computed using the formula:

$\vec{C} = det \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ as long as the components, (a_1, a_2, a_3) and (b_1, b_2, b_3) are both expressed in the $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ coordinate system.

Problems

1. Compute the determinant of $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$.

2. Compute the determinant of $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

3. Compute the determinant of $\begin{bmatrix} 3 & 4 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix}$ using co-factors.

4. Compute $\begin{bmatrix} 3 & 4 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

5. Compute $\begin{bmatrix} 3 & 4 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 1 \\ 0 & -3 & 1 \end{bmatrix}$.

6. Compute $\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix}$.

7. Compute $\begin{bmatrix} 3 & 4 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$. What kind of thing is each row of the resulting matrix?

8. Compute $\begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix}$. What kind of thing is each row of the resulting matrix?

9. For $\vec{A}=2\hat{e}_1-\hat{e}_2+3\hat{e}_3$ and $\vec{B}=-3\hat{e}_1+2\hat{e}_2+\hat{e}_3$ **, c**ompute the cross product using the determinant and co-factors.

10. If $\vec{A}=2\hat{i}-\hat{j}+3\hat{k}$ and $\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}=\begin{bmatrix} 0.5 & 0.866 & 0 \\ -.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$, what is $\vec{A}$ in the $(\hat{e}_1,\hat{e}_2,\hat{e}_3)$ coordinate system?

11. If $\vec{B}=-3\hat{e}_1+2\hat{e}_2+\hat{e}_3$ and $\vec{A}$ is given in problem 10, what is $\vec{A}\times\vec{B}$ in the $(\hat{e}_1,\hat{e}_2,\hat{e}_3)$ coordinate systems? In the $(\hat{i},\hat{j},\hat{k})$ coordinate system?