Linear Algebra

A matrix is an array of numbers arranged in a particular order, usually organized in rows and columns.

Brackets delineate the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{is a } 3 \times 3 \text{ matrix}$$

row column

Multiplying matrices proceeds by multiplying the i^{th} row on the left matrix by the j^{th} column on the right matrix to get the ij entry in the result

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{11} = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} = a_{11} b_{11} + a_{12} b_{12}$$

$$c_{21} = \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = a_{21} b_{11} + a_{22} b_{21}$$

Order matters $[A][B] \neq [B][A]$

Size matters!

The number of rows of the left matrix must match the number of columns of the right matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{23} \end{bmatrix}$$ works

$$c_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$ doesn't

$$c_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}\underbrace{(?)}_{\text{no entry}}$$

Use matrices to solve systems of equations

$$a_{11}x_1 + a_{12}x_2 = y_1$$
$$a_{21}x_1 + a_{22}x_2 = y_2$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{2\times 1}$$

$$[A][x] = [y]$$

note: # of rows of $[y]$ will be equal to # of rows of $[A]$

# of cols of $[y]$ will be equal to # of cols of $[x]$

[A] must be rectangular (# rows = # cols) and non-singular for a solution to exist.

Two special numbers ...

additive identity (ϕ) $\quad 4 + \phi = 4$

multiplicative identity (1) $\quad (4)(1) = 4$

$\cancel{4 + \phi = 4}$

$\cancel{(4)(1) =}$

Division by the additive identity is undefined for real numbers.

Same is true for matrices.

An inverse is a number that multiplies another number and yields the multiplicative identity

$$(4)^{-1}(4) = 1 \qquad \left(4^{-1} = \tfrac{1}{4}\right)$$

This gives rise to "division".

In matrices, it's more complicated.

$$[A]^{-1}[A][x] = [A]^{-1}[y]$$

$[A]^{-1}[A] = [I]$, the Identity Matrix $\left([I]_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

Show that $[A][I] = [I][A] = [A]$

A <u>singular matrix</u> does <u>not</u> have a multiplicative inverse.

For instance

$$\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$$x_1 = y_1$$
$$x_1 = y_2$$

① there is no info to give us x_2

② if $y_1 \neq y_2$, x_1 can't equal x_1!

In this class, we will avoid this type of matrix

$$[A]^{-1} = \frac{[A]^{+}}{|A|}$$

$[A]^{+}$ is the adjugate (formerly classical adjoint or adjunct) matrix.

$[A]^{+} = [C]^{T}$ where $[C]$ is the <u>transpose</u> of the matrix of co-factors

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$[A]^{T}$ is formed by swapping rows & cols of $[A]$

$$\begin{bmatrix}\boxed{a_{11} \quad a_{12}}\\ \boxed{a_{21} \quad a_{22}}\end{bmatrix}^{T} = \begin{bmatrix}\boxed{\begin{matrix}a_{11}\\ a_{12}\end{matrix}} & \boxed{\begin{matrix}a_{21}\\ a_{22}\end{matrix}}\end{bmatrix}$$

A co-factor is a real number formed by evaluating the <u>determinant</u> of a sub-matrix which is formed by deleting the row & col of the larger matrix

For

$$\begin{array}{cc} & \text{col 1} \\ \text{row 1} & \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13}} \\ a_{21} \quad a_{22} \quad a_{23} \\ a_{31} \quad a_{32} \quad a_{33} \end{bmatrix} \end{array}$$

$$C_{11} = (-1)^{\overset{\text{row + col}}{1+1}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

The determinant of a matrix is a real number formed using co-factors. (det of singular matrix is zero)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

The determinant of a 2x2 matrix can be determined from formula:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} C_{11} + a_{12} C_{12}, \quad C_{11} = (-1)^{1+1} a_{22} \quad C_{12} = (-1)^{1+2} a_{21}$$

Therefore, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (formula)

Back to 3x3 det

$$a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= +a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

2x2 inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{[A]^{+}}{|A|} = \frac{[A]^{+}}{(ad-bc)} = \frac{\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^{T}}{ad-bc}$$

$$\left.\begin{aligned} C_{11} &= (-1)^{1+1}\, d = d \\ C_{12} &= (-1)^{1+2}\, c = -c \\ C_{21} &= (-1)^{2+1}\, b = -b \\ C_{22} &= (-1)^{2+2}\, a = a \end{aligned}\right\}$$

$$= \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$$ (formula)

3x3 & higher matrices are a beast no matter how you slice it. Let's do an example

$$[A]^{-1} = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 4 & 0 & 2 \end{bmatrix}^{-1}$$

$$[C] = \begin{bmatrix} (-1)^2(-2) & (-1)^{1+2}(2-4) & (-1)^{1+3}(0-(-4)) \\ (-1)^{2+1}\,4 & (-1)^{2+2}(6) & (-1)^{2+3}(-8) \\ (-1)^{3+1}\,2 & (-1)^{3+2}(3) & (-1)^{3+3}(-3-2) \end{bmatrix} =$$

$$\begin{bmatrix} -2 & 2 & 4 \\ -4 & 6 & 8 \\ 2 & -3 & -5 \end{bmatrix}$$

$$|A| = (3)(-2) + 2(2) + 0(4) = -2$$

$$[A]^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 3/2 \\ -2 & -4 & 5/2 \end{bmatrix}$$

woo-hoo!

Check $[A]^{-1}[A] = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 3/2 \\ -2 & -4 & 5/2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 3/2 \\ -2 & -4 & 5/2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \\ 14 \end{bmatrix}$$

check

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} -6 \\ 8 \\ 14 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \quad \text{OK}$$