Crash Course on Eigenvalues

Consider a square matrix $[A] = [[A_1][A_2][A_3] \cdots [A_n]]$

where $[A_i]$ are column vectors.

$$\text{E.g.} \quad [A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \Bigg[\underbrace{\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}}_{\text{each } 3\times1} \Bigg]$$

3 vectors

In other words, an $n \times n$ matrix of scalars is a $1 \times n$ matrix of column vectors of size $n \times 1$

The product of $[A][x]$ between an $n\times n$ matrix and an $n\times 1$ column vector $[x] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ ← scalars, is the sum,

$$[A][x] = x_1[A_1] + x_2[A_2] + \cdots + x_n[A_n]$$

So, $[A][x] = [b]$ can be seen to be

$$x_1[A_1] + x_2[A_2] + \cdots + x_n[A_n] = [b]$$

So, $[b]$ is a linear combination of the set $\{[A_1], [A_2], \cdots, [A_n]\}$. In other words, $[A]_i$ span a space. If $[b]$ is a member of that space, then there is a sol'n. If not, then there is no sol'n.

If $[A_i]$ span $n \times 1$ vector space, then a soln exists.

If $\exists$ not a set of non-zero numbers $a_1 \cdots a_n \ni$

$$a_1[A_1] + a_2[A_2] + \cdots + a_n[A_n] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

then $[A_i]$ are linearly <u>dependent</u>.

E.g. $[A] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow (1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - (1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

linear independent $\Rightarrow$ only $\{a_1, a_2, \ldots, a_n\}$ for which
$a_1[A_1] + a_2[A_2] + \cdots + a_n[A_n] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ are $a_1 = a_2 = \cdots = 0$.

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if $[A_i]$ are linearly dependent, $\exists$ infinite soln's to
$[A][x] = [b]$ & $[A]$ is called <u>singular</u>
if $[A_i]$ are linearly independent, $\exists$ 1 soln to
$[A][x] = [b]$ & $[A]$ is called non singular.

For two columns of $[A]$, $[A_i]$ & $[A_j]$, can form a scalar product as

$$[A_i]^T[A_j] = [A_j]^T[A_i] = S$$

if $S = 0$ for, $i \neq j$ and $\neq 0$ for $i = j$,
then $[A_i]$ are called orthogonal.

if $[A_i]^T[A_i] = 1$, columns are called orthonormal.

If $[A_i]$ are orthonormal & linearly independent, they form an <u>orthonormal</u> <u>basis</u>.

For a matrix $[A]$, are there any numbers, λ, for which

$$[A][x] = \lambda[x] \ ?$$

Call λ (if they exist ... and they do) an eigenvalue.
Call all $[X]$ that satisfy the eqn. eigenvectors.

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eigen is German for "strange" or "special".

Since $[x] = [I][x]$,

$$[A][x] = \lambda[x] \Longrightarrow$$

$$([A] - \lambda[I])[x] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longleftarrow$$

What λ make $[A]-\lambda[I]$ singular?

Without proof, $\det([A]) = \phi \Rightarrow [A]$ is singular.

So, solve $\det([A]-\lambda[I]) = \phi$ to find λ.

It can be shown that $\det([A]-\lambda[I]) = p(\lambda)$ is an n^{th} order polynomial in λ. $\therefore \exists$ n eigenvalues (not necessarily distinct) for $[A]$ (Fund. Thm. of Algebra).

Note: $\lambda = \phi \Rightarrow$ the original matrix was already singular.

Example:

$$[A] = \begin{bmatrix} -1 & 6 & 2 \\ 0 & 5 & -6 \\ 1 & 0 & -2 \end{bmatrix}$$

$$[A] - \lambda[I] = \begin{bmatrix} -1-\lambda & 6 & 2 \\ 0 & 5-\lambda & -6 \\ 1 & 0 & -2-\lambda \end{bmatrix}$$

$$\det([A] - \lambda[I]) = (-1-\lambda)\begin{vmatrix} 5-\lambda & -6 \\ 0 & -2-\lambda \end{vmatrix} - (6)\begin{vmatrix} 0 & -6 \\ 1 & -2-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & 5-\lambda \\ 1 & 0 \end{vmatrix} =$$

$$(-1-\lambda)(5-\lambda)(-2-\lambda) - 6(6) + 2(-5+\lambda)$$

$$= p(\lambda)$$

$$p(\lambda) = -\lambda^3 + 2\lambda^2 + 15\lambda - 36 \Rightarrow \lambda = -4, 3$$

$\lambda = -4$

$$\begin{bmatrix} 3 & 6 & 2 \\ 0 & 9 & -6 \\ \cancel{1} & \cancel{0} & \cancel{2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{r} 18 \\ 36 \\ \hline 54 \end{array}$$

~~0 6 -4~~

$9① - 6②$ 27   0   54 $\rightarrow$ 1   0   2

$$\Rightarrow \left.\begin{array}{l} x_1 + 2x_3 = 0 \\ 9x_2 - 6x_3 = 0 \end{array}\right\} \Rightarrow \text{let } x_3 = x$$

$$\begin{bmatrix} -2x \\ 2/3\,x \\ x \end{bmatrix} = \begin{bmatrix} -2 \\ 2/3 \\ 1 \end{bmatrix} x$$

$$\lambda = 3 \quad \begin{bmatrix} -4 & 6 & 2 \\ 0 & 2 & -6 \\ 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3(2) - (1) \quad \begin{matrix} 4 & 0 & -20 & \rightarrow \\ 1 & 0 & -5 & \end{matrix}$$

$$\begin{aligned} 2x_2 - 6x_3 &= \phi \\ x_1 - 5x_3 &= \phi \end{aligned} \qquad x_3 = x \Rightarrow$$

$$[X] = \begin{bmatrix} 5x \\ 3x \\ x \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} x$$

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normalize by setting $x = \left(\sqrt{[X]^T[X]}\right)^{-1}$

$$[x_1] = \begin{bmatrix} -2 \\ 2/3 \\ 1 \end{bmatrix} \frac{1}{\sqrt{4 + \frac{4}{9} + 1}}$$

$$[x_2] = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \frac{1}{\sqrt{25 + 9 + 1}}$$

Two distinct eigenvalues $\Rightarrow$ rank$(A) = 2$

If all eigenvalues are distinct, form a matrix of normalized eigenvectors,

$$[S] = [\underbrace{[x_1]\,[x_2] \cdots [x_n]}_{\text{eigenvectors}}] \Rightarrow S^{-1} A S = D, \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Applications $\rightarrow$ principal stresses, principal moments of inertia.

FYI, symmetric matrices have distinct eigenvalues.

If eigenvalues are distinct, eigenvectors are independent.

Consider non singular matrix S. S^{-1} exists &

$S S^{-1} = S^{-1} S = I$.

$(A - \lambda I)[x] = \phi$

$(A - \lambda S^{-1} S)[x] = \phi$

$((S^{-1}S) A (S^{-1}S) - \lambda S^{-1} S)[x] = \phi$

$S^{-1}(SAS^{-1} - \lambda [I]) S [x] = \phi$

S^{-1}, S are non singular $\Rightarrow$ $SAS^{-1} - \lambda [I]$ is singular

λ is eigenvalue of $[A]$ iff λ is eigenvalue of SAS^{-1}

A & SAS^{-1} are <u>similar</u>.

Assume eigenvectors of $[A]$ are lin. ind. $\Rightarrow$

Let $S = [[x_1][x_2] \cdots [x_n]]$

$[A][x_i] = \lambda_i [x_i]$ (defn of $[x_i]$)

$[A][S] = [A[x_1] \; A[x_2] \cdots A[x_n]] = [\lambda_1[x_1] \; \lambda_2[x_2] \cdots \lambda_n[x_n]]$

$[S]^{-1}[A][S] = [\lambda_1 [S]^{-1}[x_1] \;\; \lambda_2 [S]^{-1}[x_2] \cdots \lambda_n [S]^{-1}[x_n]]$

Note $[S]^{-1}[S] = [[S]^{-1}[x_1] \;\; [S]^{-1}[x_2] \cdots [S]^{-1}[x_n]] = [I]$

Hence $[S]^{-1}[x_1] = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $[S]^{-1}[x_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, etc.

so $[S]^{-1}[A][S] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$ = diagonal.

If $[A]$ has lin. ind. eigenvecs, then $[A]$ is similar to diagonal matrix.

If $[A]$ is similar to diagonal matrix, then

$$[C]^{-1}[A][C] = [D] = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$\Rightarrow$ $[A][C] = [C][D]$ columns of $[C]$

$[C] = [[C_1]\ [C_2]\ \cdots\ [C_n]]$

$$\Big[\underbrace{[A][C_1]}_{\text{vector}}\ \ [A][C_2]\ \cdots\ [A][C_n]\Big] = \big[d_1[C_1]\ \ d_2[C_2]\ \cdots\ d_n[C_n]\big]$$

$\Rightarrow$ $[A][C_1] = d_1[C_1]$

$[A][C_2] = d_2[C_2]$

$\vdots$

$[A][C_n] = d_n[C_n]$

C_i are linearly ind. & d_i are the eigenvalues of A (by hypothesis)

$\Rightarrow$ C_i are the eigenvectors of $[A]$

Do example via matlab