Diagonalization of a Symmetric Matrix

Symmetric matrices appear in all phases of engineering. Knowing some of their special properties can help an engineer maximize the effects of symmetry.

Consider a $n\times n$ symmetric matrix $[A]$.

This matrix will have $1\leq k\leq n$ distinct eigenvalues. (Fundamental Theorem of Algebra)

1. Orthogonality of eigenvectors of a symmetric matrix
The eigenvectors corresponding to distinct eigenvalues are orthogonal.
An eigenvector satisfies $[A]\vec{u}_1=\lambda_1\vec{u}_1$ and multiplying by $\vec{u}_2^T$ yields $\vec{u}_2^T[A]\vec{u}_1=\lambda_1\vec{u}_2^T\vec{u}_1$

An eigenvector satisfies and $[A]\vec{u}_2=\lambda_2\vec{u}_2$ and multiplying by $\vec{u}_1^T$ yields $\vec{u}_1^T[A]\vec{u}_2=\lambda_2\vec{u}_1^T\vec{u}_2=\lambda_2\vec{u}_2^T\vec{u}_1$.
Since $(\vec{u}_1^T[A]\vec{u}_2)^T=\vec{u}_2^T[A]^T\vec{u}_1=\vec{u}_2^T[A]\vec{u}_1$ (by the symmetry of $[A]$), $\lambda_1\vec{u}_2^T\vec{u}_1=\lambda_2\vec{u}_2^T\vec{u}_1$ which can only occur if $\vec{u}_2^T\vec{u}_1=0$, proving that the eigenvectors corresponding to distinct eigenvalues in a symmetric matrix are orthogonal.

2. Orthogonal Complement to a symmetric matrices eigensubspace
Consider a set of eigenvectors corresponding to a set of distinct eigenvalues of a symmetric matrix, $[A]$, $\{\vec{u}_{1,}\vec{u}_{2,}\ldots,\vec{u}_k\}$ where $1\leq k<n$ (in other words, the number of eigenvectors is less than the size of the matrix)

Define a subspace, $W^{\perp}=\{\vec{x}:\vec{x}^T\vec{u}_i=0,i=1\ldots k\}$. This space is called the "orthogonal complement" to the set of eigenvalues. In other words, it is the set of all things orthogonal to all of those eigenvalues.

Since the set of eigenvectors is less than n, there is at least one non-zero vector in $\mathbb{R}^n$ that is also in $W^{\perp}$. (Basis for $\mathbb{R}^n$ consists for n vectors.) As a result, the dimension of $W^{\perp}$ is also greater than or equal to 1.

For a member $\vec{x}\in W^{\perp}$, $\vec{x}^T\vec{u}_i=0$. Since $\vec{u}_i$ is an eigenvector of $[A]$, $[A]\vec{u}_i=\lambda_i\vec{u}_i$.
Multiplying $\vec{x}^T\vec{u}_i=0$ by λ_i does not change anything (even if $\lambda_i=0$), so $\lambda_i\vec{x}^T\vec{u}_i=0=\vec{x}^T(\lambda_i\vec{u}_i)=\vec{x}^T([A]\vec{u}_i)=(\vec{x}^T[A])\vec{u}_i=([A]^T\vec{x})^T\vec{u}_i=([A]\vec{x})^T\vec{u}_i$.

This shows that, for $\vec{x}\in W^{\perp}$, $[A]\vec{x}\in W^{\perp}$.

The set, $W^{\perp}$, is a subspace of $\mathbb{R}^n$, since $\vec{x},\vec{y}\in W^{\perp}$ means that $\vec{x}+\vec{y}\in W^{\perp}$ and since $\vec{x}\in W^{\perp}$ means that $a\vec{x}\in W^{\perp}$, where a is a non-zero scalar. These properties are easily shown.

3. **Minimum Annihilating Polynomial**
Consider $\vec{x}_0\in W^{\perp}$ where $\vec{x}_0\neq\hat{0}$. Then, $[A]\vec{x}_0\in W^{\perp}$. If $[A]\vec{x}_0\in W^{\perp}$, then $[A]^2\vec{x}_0\in W^{\perp}$.
And so on, such that $[A]^k\vec{x}_0\in W^{\perp}$. Since $\{\vec{x}_{0,}[A],\ldots,[A]^k\vec{x}_0\}\in W^{\perp}$, there exists $k\leq dim(W^{\perp})+1$ such that this set of vectors is linearly dependent (Theorem 2.3 from Johnson,

Reiss). Therefore, there exist unique scalars such that $a_0 \vec{x}_0 + a_1 [A] + ... + a_k [A]^k \vec{x}_0 = \hat{0}$.

Define a polynomial, $m(t) = a_0 + a_1 t + ... + a_k t^k$, then $m([A]) \vec{x}_0 = \hat{0}$. This polynomial is called the "minimum annihilating polynomial" and the order of m is the smallest order that can result in $m([A]) \vec{x}_0 = \hat{0}$.

If r is a root of m(t), then $m(t) = (t - r) s(t)$. The order of s(t) is less than k, which means it cannot be a "minimum annihilating polynomial" and $\vec{u} = s([A]) \vec{x}_0 \neq \hat{0}$.

Hence, $m([A]) \vec{x}_0 = ([A] - r[I]) \vec{u} = \hat{0}$, which implies that $[A] \vec{u} = r \vec{u}$, which means that r is an eigenvalue of $[A]$ and $\vec{u} = s([A]) \vec{x}_0$ is an eigenvector of $[A]$.

4. **Symmetric Matrix has Full Complement of Orthogonal Eigenvectors**
Since $\vec{x}_0 \in W^{\perp}$, $\vec{u} = s([A]) \vec{x}_0 \in W^{\perp}$. This eigenvector is perpendicular to the other eigenvectors, $\{\vec{u}_{1,} \vec{u}_{2,} ..., \vec{u}_k\}$, that defined $W^{\perp}$.

So, there is now a set of k+1 orthogonal eigenvectors, $\{\vec{u}_{1,} \vec{u}_{2,} ..., \vec{u}_k, \vec{u}\}$.

Rinse and repeat until the set contains all n eigenvectors of $[A]$.

Therefore, there exist n orthogonal eigenvectors for a symmetric matrix.

5. **Similarity:** Form a matrix from the n eigenvectors of $[A]$, $[S] = [\vec{u}_1 \vec{u}_2 ... \vec{u}_n]$. This matrix's transpose is its inverse, $[S]^T = [S]^{-1}$.

This matrix diagonalizes $[D] = [S]^T [A][S]$.

Since the diagonal matrix will be similar to the original symmetric matrix, the eigenvalues will be the same and will be the diagonal entries.

6. **Construction**:
   a. Determine the distinct eigenvalues of the matrix.
   b. Determine the eigenvector for those eigenvalues.
   c. Determine additional vectors per the construction in step 3.
   d. Form a matrix with each of the vectors determined in b, c as columns.
   e. Form the inverse of the matrix in 6.d by taking its transpose.
   f. Form $[D] = [S]^T [A][S]$.